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Even door problem

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Abstract

There are several methods of π estimation. In this research, we adjusted some conditions base on the research of Alberto Zorzi [1]. The number of doors has been changed into an even number as well as increased the number of cars. Due to this adjustment, the winning probability of the player will be double when the number of doors is even and there are two cars hidden inside, compare with the situation that the number of doors is even and there is one car hidden inside. If the number of doors is even and there are 3 cars or more, we will not be able to play this game with the same strategy anymore. Moreover, this research demonstrates the relationship between the probabilities to win the game with the value of π that is the player can estimate the value of π by playing game repeatedly with the best strategy.

Keywords: *pi, Monty Hall Problem, n doors.*

1. Introduction

Monty Hall problem is a well-known paradox problem. It is named after Monty Hall, the host of the show Let’s Make A Deal. This is how to play the game, there are 3 closed doors, one of the doors has a car inside and the rest of them have sheep inside. The host knows what is behind each door but the player does not know. When the player selects one door, the host will open the other door which has no car inside. The host will ask the player whether the player wants to reselect the door or not. The player will win the game by selecting the door that has the car inside.

The problem is which solution is the best between stick with the old door or reselect the new one. If the player reselects the door, the probability to win is $\frac{2}{3}$ because if they reselect the door and found the car means that the first door that they selected has sheep inside. The probability to select the door that has sheep inside at the first place is $\frac{2}{3}$. It is crystal clear that if the player decided not to reselect the door, the chance to get the car is $\frac{1}{3}$ which contradicts with people’s feelings because they think that there are two closed doors so the probability to find the car is $\frac{1}{2}$ no matter they change the door or not. For mathematical proof, there are several ways to solve this classic problem, see [2] for examples.

Moreover, many researchers have extended this problem in several ways such as changing the rule of game [3], increasing the number of players [4], or applying to the other fields [5, 6].

In 2009, Alberto Zorzi [1] magnifies the problem by setting up the number of closed doors as n when n is an odd number which is greater than or equal to 3 and has only one door which got the car inside. He proved that if we substitute the number of doors as n and n is an odd number which is more than or equal to 3, the best solution for the player is to reselect the door whenever the host asks. In addition, he showed that

$P'(n)=\frac{2\cdot 4\cdot 6\cdots(n-3)(n-1)}{3\cdot 5\cdot 7\cdots(n-2)n}$ and $\lim_{n\to\infty}\left[P'(n)\right]^2\cdot n=\frac{\pi}{2}$ where the probability that the player will find the car by letting the player reselects the door whenever the host asks denoted by $P'(n)$.

In this research, we will find the best solution under the condition that the number of doors is an even number which is greater than or equal to 4 and there are 2 cars hidden inside. This proof is showed in theorem 2.

Problem 1: Even Doors, One Car

There are n closed doors where n is an even number which is greater than or equal to 4. There is one car hidden inside and the rest of the doors have sheep inside. Only the host knows which door has the car inside and the player has to guess to win the game. The game will begin and play the same way as Alberto Zorzi's [1] and it will over when there is one closed door which has not been selected and the last door that has been selected will be opened.

For the problem 1, we let $P(n)$ be the probability that the player will find the car when the player reselects the doors whenever the host asks and $F(n,m)$ substitutes the probability that the player will find the car if the player reselects the door $m-1$ times, $1\le m\le\frac{n}{2}-1$, when the host asks. For the next time, the player will not reselect the doors.

Theorem 1

From the problem 1, the best solution for the player is to reselect the door whenever the host asks. Furthermore,

1) $P(n)=\frac{1\cdot 3\cdot 5\cdots(n-3)(n-1)}{2\cdot 4\cdot 6\cdots(n-2)n}$ when n is an even number which is greater than or equal to 4.

2) $F(n,1)=\frac{1}{n}$ when n is an even number which is greater than or equal to 4.

3) $F(n,m)=\frac{(n-1)(n-3)(n-5)\cdots(n-2m+3)}{n(n-2)(n-4)\cdots(n-2m+4)}\cdot\frac{1}{n-2m+2}$ when n is an even number which is greater than or equal to 6 and $2\le m\le\frac{n}{2}-1$.

Proof

1). Similar to theorem 1 in [1], we can prove this statement.

2). Since the probability that the player will find the car if the player does not reselect when the host asks for the first time equal to $\frac{1}{n}$, we have $F(n,1)=\frac{1}{n}$ and $P(n)=\frac{1\cdot 3\cdot 5\cdots(n-3)(n-1)}{2\cdot 4\cdot 6\cdots(n-2)n}>\frac{1}{n}$.

In conclusion, the probability that the player will find the car if the player reselect the door whenever the hosts asks more than the probability that the player will find the car if the player does not reselect whenever the hosts asks.

3). It is easy to see that $F(n,2)=\frac{n-1}{n}\cdot\frac{1}{n-2}$. Let $F(n,m)=\frac{(n-1)(n-3)\cdots(n-2m+3)}{n(n-2)\cdots(n-2m+4)}\cdot\frac{1}{n-2m+2}$ where n is an even number which is greater than or equal to 6 and $F(n,2)=\frac{n-1}{n}\cdot\frac{1}{n-2}$. Assume that the number of door as $n+2$ doors and there is one door that has a car inside, if the player reselects the door m times and will not reselect whenever the hosts asks, which means that the door which the player selected has nothing hidden inside, so the probability is $\frac{n+1}{n+2}$. After that when the hosts choose to open the door which has no car and the player did not choose. We can do it n choice. The probability of each choice is $\frac{1}{n}$ because the player decides to reselect the door this time, there are n doors to select and the player will reselect as $m-1$ times and next time, the player will not reselect the door so the probability is $F(n,m)$.

Hence, the probability that the player will find the car if the player does not reselect the door is m times whenever the host asks and the player will not reselect the door for the next time equals to $\frac{(n+1)}{(n+2)} \cdot n \cdot \frac{1}{n} \cdot F(n,m) \cdot$

Therefore

$$F(n+2,m+1)=\frac{n+1}{n+2}\cdot\frac{(n-1)(n-3)(n-5)\cdots(n-2m+3)}{n(n-2)(n-4)\cdots(n-2m+4)}\cdot\frac{1}{n-2m+2} \quad =\frac{(n-1)(n-3)\cdots(n-2m+3)}{n(n-2)\cdots(n-2m+4)}\cdot\frac{1}{n-2m+2}.$$

By mathematical induction, we get

$$F(n,m)=\frac{(n-1)(n-3)\cdots(n-2m+3)}{n(n-2)\cdots(n-2m+4)}\cdot\frac{1}{n-2m+2}$$

Since n is an even number which is greater than or equal to 6 and $2 \le m \le \frac{n}{2}-1$, we have

$$\frac{(n-1)(n-3)\cdots(n-2m+3)}{n(n-2)\cdots(n-2m+4)}\cdot\frac{1}{n-2m+2} < \frac{(n-1)(n-3)\cdots(n-2m+3)}{n(n-2)\cdots(n-2m+4)}\cdot P(n-2m+2)=P(n).$$

Hence, the probability that the player will find the car if the player reselect the door whenever the host asks is higher than the probability that the player decided not to reselect the door every m times that the host asks.
In conclusion from 1)-3), if the number of door is n when n is an even number which is greater than or equal to 4 the perfect solution for the player is to reselect the door whenever the host asks.

Problem 2: Even Doors, Two Cars

There are n closed doors where n is an even number which is greater than or equal to 4. There are 2 doors which have the cars inside; the rest of the doors have sheep behind. The host knows which door has the car. But the player does not know and have to guess the right door to win the game. This game plays the same way as the problem 1.
In problem 2, we denote $Q(n)$ as the probability that the player will find the car by reselecting the door whenever the host asks and $G(n,m)$ the probability that the player will find the car if the player reselect the door $m-1$ times, $1 \le m \le \frac{n}{2}-1$, when the host asks. For the next time, the player decided not to reselect the door.

Theorem 2

In the problem 2, the perfect solution for the player is to reselect the door whenever the host asks. So we will get

1) $Q(n)=\frac{3\cdot 5\cdots(n-3)(n-1)}{4\cdot 6\cdots(n-2)n}$ when n is an even number which is greater than or equal to 4.

2) $G(n,1)=\frac{2}{n}$ when n is an even number which is greater than or equal to 4.

3) $G(n,m)=\frac{(n-1)(n-3)(n-5)\cdots(n-2m+3)}{n(n-2)(n-4)\cdots(n-2m+4)}\cdot\frac{2}{n-2m+2}$ where n is an even number which is greater than or equal to 6 and $2 \le m \le \frac{n}{2}-1.$

Proof

1). Assume that the player reselects the door whenever the host asks. Clearly, $Q(4)=\frac{3}{4}$.

Assume that $Q(n)=\frac{3\cdot 5\cdots(n-3)(n-1)}{4\cdot 6\cdots(n-2)n}$ where n is an even number which is greater than or equal to 4.

To show that $Q(n+2)=\dfrac{3\cdot 5\cdots(n-3)(n-1)(n+1)}{4\cdot 6\cdots(n-2)n(n+2)}$, we assume that the number of door is $n+2$ and there are 2 doors which have the car inside. We will analyze these 2 cases.

<u>Case 1</u> The probability that the player select the door which has no car inside equal to $\dfrac{n}{n+2}$, the host opened the door that the player did not choose and has no car inside. We can do it $n-1$ choice. The probability of each choice equal to $\dfrac{1}{n-1}$ due to the player decided to reselect the door whenever the host asks and the meantime the number of available door is n. There are two doors among those available doors have the car inside. It means that the probability that the player will find the car when the player reselect the door whenever the host asks equal to $Q(n)$. Therefore, in this case the probability that the player will find the car if the player reselects the door whenever the host asks is equal to

$$\frac{n}{n+2}\cdot(n-1)\cdot\frac{1}{(n-1)}\cdot Q(n)=\frac{n}{n+2}\cdot\frac{3\cdot 5\cdots(n-3)(n-1)}{4\cdot 6\cdots(n-2)n}.$$

<u>Case 2</u> The probability that the player will find the car from the beginning is equal to $\dfrac{2}{n+2}$, the host opened unselected door which has no car inside. We can do it n ways, the probability of each way equal $\dfrac{1}{n}$. From theorem 1, the result for this case is the probability that the player will find the car if the player reselect the door whenever the host asks is equal to

$$\frac{2}{n+2}\cdot n\cdot\frac{1}{n}\cdot P(n)=\frac{2}{n+2}\cdot\frac{1\cdot 3\cdot 5\cdots(n-3)(n-1)}{2\cdot 4\cdot 6\cdots(n-2)n}.$$

From these 2 cases, the probability that the player will find the car when the player reselects the door whenever the host asks equals to

$$\frac{n}{n+2}\cdot\frac{3\cdot 5\cdots(n-3)(n-1)}{4\cdot 6\cdots(n-2)n}+\frac{1}{n+2}\cdot\frac{3\cdot 5\cdots(n-3)(n-1)}{4\cdot 6\cdots(n-2)n}=\frac{3\cdot 5\cdots(n-3)(n-1)(n+1)}{4\cdot 6\cdots(n-2)(n)(n+2)}.$$

Then $Q(n+2)=\dfrac{3\cdot 5\cdots(n-3)(n-1)(n+1)}{4\cdot 6\cdots(n-2)(n)(n+2)}$. By mathematical induction, we have

$$Q(n)=\frac{3\cdot 5\cdots(n-3)(n-1)}{4\cdot 6\cdots(n-2)n}$$

where n is an even number which is greater than or equal to 4.

2). Due to the probability that the player will find the car if the player decided not to reselect the door at the first time that the host asked is equal to $\dfrac{2}{n}$. Hence, $G(n,1)=\dfrac{2}{n}$ and $Q(n)\geq\dfrac{2}{n}=G(n,1)$.

Therefore, the probability that the player will find the car if the player reselect the door whenever the host asks is higher than the probability that the player will find the car if the player did not reselect the door when the host asked for the first time.

It is clear that $G(n,2)=\dfrac{n-1}{n}\cdot\dfrac{2}{n-2}$.

Assume that $G(n,m)=\dfrac{(n-1)(n-3)(n-5)\cdots(n-2m+3)}{n(n-2)(n-4)\cdots(n-2m+4)}\cdot\dfrac{2}{n-2m+2}$ when n is an even number which is greater than or equal to 6 and m is counting number by $2\leq m\leq\dfrac{n}{2}-1.$

To show that $G(n+2,m+1)=\dfrac{(n+1)(n-1)(n-3)\cdots(n-2m+3)}{(n+2)n(n-2)\cdots(n-2m+4)}\cdot\dfrac{2}{n-2m+2}$, we assume that the number of doors are $k+2$ and the player does not reselect the door $t+1$ times when the host asks. Among k doors, there are 2 doors which have the car inside. We will analyze these 2 cases.

<u>Case 1</u> The probability that the player selects the door which has no car inside from the first time is equal to $\dfrac{k}{k+2}$, when the host opened the unselected door which has no car inside, we can do it $k-1$ ways. The probability of each way equals to $\dfrac{1}{k-1}$ due to the player reselect the door this time, there are k doors to select and the player

does not reselect the door t times when the host asks. Among k doors, there are 2 doors which have the cars inside. The probability that the player will find the car is equal to

$$G(k,t)=\frac{(k-1)(k-3)\cdots(k-2t+3)}{k(k-2)\cdots(k-2t+4)}\cdot\frac{2}{k-2t+2}.$$

Hence, the probability that the player will find the car is $\frac{k}{k+2}\cdot(k-1)\cdot\frac{1}{(k-1)}\cdot G(k,t)\cdot$

<u>Case 2</u> The probability that the player choose the door which has the car inside from the first time equals to $\frac{2}{k+2}$, when the host opened the unselected door which has no car inside and the player did not choose, we can do it k ways. The probability of each way equals to $\frac{1}{k}$. Due to this time the player decided to reselect the door and there are k doors available to select. The player will not reselect the door at t times when the host asks. Among k doors, there is one door which has the car inside. From Theorem 1 the probability that the player will find the car is $F(k,t)=\frac{(k-1)(k-3)\cdots(k-2t+3)}{k(k-2)\cdots(k-2t+4)}\cdot\frac{1}{k-2t+2}.$

Hence, in this case the probability that the player will find the car equals to $\frac{2}{k+1}\cdot(k)\cdot\frac{1}{(k)}\cdot F(k,t).$

From these 2 cases, we will get

$$G(k+2,t+1)=\frac{k}{k+2}\cdot G(k,t)+\frac{2}{k+2}\cdot F(k,t)$$

$$=\frac{(k+1)(k-1)\cdots((k+2)-2(t+1)+3)}{(k+2)k\cdots((k+2)-2(t+1)+4)}\cdot\frac{2}{(k+2)-2(t+1)+2}.$$

By mathematical induction, we have $G(n,m)=\frac{(n-1)(n-3)(n-5)\cdots(n-2m+3)}{n(n-2)(n-4)\cdots(n-2m+4)}\cdot\frac{2}{n-2m+2}.$

Note that $\frac{(n-1)(n-3)\cdots(n-2m+3)}{n(n-2)\cdots(n-2m+4)}\cdot\frac{2}{n-2m+2}<\frac{(n-1)\cdots(n-2m+3)}{n\cdots(n-2m+4)}\cdot Q(2n-2m+2)=Q(n).$

Hence, the probability that the player will find the car if the player reselect the door whenever the host asks is more than the probability that the player will find the car if the player choose not to reselect the door m times when the host asks

In the conclusion, from 1) -3), if the number of doors is n and n is an even number which is greater than or equal to 4. The perfect solution for the player is to reselect the door whenever the host asks.

From Theorem 2, we see that it cannot play with the same rule if the number of cars is three or more than three. Consequently, by Theorem 2 and Wallis's Formula, we will get this following theorem.

Theorem 3

If $Q(n)=\frac{3\cdot5\cdot7\cdots(n-1)}{4\cdot6\cdot8\cdots n}$ when n is an even number which is more than or equal to 4. Hence,

$$\lim_{n\to\infty}\left[Q(n)\right]^2\cdot(n+1)=\frac{8}{\pi}.$$

Proof

Let $Q(n)=\frac{3\cdot5\cdot7\cdots(n-1)}{4\cdot6\cdot8\cdots n}$ when n is an even number which is more than or equal to 4. Let $n=2m$ for some integer number m which is more than or equal to 2. Then

$$\left[Q(n)\right]^2\cdot(n+1)=\left[\frac{3\cdot5\cdot7\cdots(n-1)}{4\cdot6\cdot8\cdots n}\right]^2\cdot(n+1)=\frac{3^2\cdot5^2\cdot7^2\cdots(2m-1)^2\cdot(2m+1)}{4^2\cdot6^2\cdot8^2\cdots(2m)^2}$$

$$= \frac{2^2}{\dfrac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdots (2m)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2m-1)^2 \cdot (2m+1)}}.$$

Thus $\lim_{n\to\infty} \left[Q(n)\right]^2 \cdot (n+1) = \lim_{m\to\infty} \frac{2^2}{\dfrac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdots (2m)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2m-1)^2 \cdot (2m+1)}}.$

By Wallis' Formula, we complete the proof.

2. Conclusion

In these generalize Monty Hall Problems, the best strategy for the player is to reselect the door whenever the host asks. The winning probability of the player will be double when the number of doors is even and there are two cars hidden inside, compare with the situation that the number of doors is even and there is one car hidden inside. However, it cannot play with the same rule if the number of cars is three or more than three. The last theorem shows that we can estimate the value of π by playing the game repeatedly using the best strategy.

3. Reference

[1] Alberto, Z., 2009. Cars, Goats, π, and e. Mathematics Magazine 82, 360-363.

[2] Joe, R., Irvin, S., The Full Monty. In the MWSUG conference. [WWW Document] URL http://www.mwsug.org/index.php/2011-proceedings.html. (accessed 23. 4. 15).

[3] Jeffrey, R., 2008. Monty Hall, Monty fall, Monty crawl. Math Horizons 16, 5-7.

[4] Stephen, L., Jason, R., Andrew, S., 2009. The Monty Hall Problem, Reconsidered. Mathematics Magazine 82, 332-342.

[5] Flitney, A.P., Abbott, D., 2008. Quantum version of the Monty Hall problem. Physical Review A 65, 062318.

[6] Hammad, S., 2009. Is the lure of choice reflected in market prices? Experimental evidence based on the 4-door Monty Hall problem. Journal of Economic Psychology 30, 203-215.