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The placement of points on a unit circle: the golden placement policy

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Abstract

This article investigates patterns and properties among the points and the gaps that are formed when points are placed in a clockwise direction around a unit circle so that the distance between successive placements is the Golden Section (τ). It is shown that the use of τ corresponds to a policy of placing points in the oldest gap among all of the largest gaps. The analyses examine the ordering of the points; the length, age, type, and the numbers of gap types formed; and demonstrate the importance of Fibonacci numbers in describing the patterns that emerge. The analyses provide insights into practical situations including the placement of transmitters and receivers and phyllotaxis.

Keywords: Fibonacci numbers; Golden section; Phyllotaxis

1. Introduction

Points are placed clockwise in succession on the circumference of a unit circle with radius $1/2\pi$ at a distance α, or an angle of $\alpha/2\pi$, from each other with the first point labeled 0.

If α is the rational number m/n then the points labeled 0, 1, 2, 3, **…**, n – 1 will appear as distinct points on the circle but when the next point labeled n is placed it will correspond exactly with the point labeled 0 and the circle will have been traversed m times. Points labeled n + 1, n + 2, n + 3, **…** will correspond exactly with those already labeled 1, 2, 3, **…**, n – 1 and the distance between any pair of adjacent points is constant at 1/n. This pattern is not of further interest and so α is restricted to be an irrational number.

Practical situations involve the placement of transmitters or receptors around the circumference of a circle. A different situation, shown in Figure 1, involves the positioning of leaf nodes on the stem of a plant (phyllotaxis) where the stem is approximated by a cylinder with leaf nodes emerging in succession at points along a spiral which winds upward around the stem of the plant. If the points representing the leaf nodes on the stem of the plant are projected onto points on the unit circle at the base of the stem then analyzing this pattern of points provides insights into the emergence of the leaf nodes on the stem of the plant. An excellent introduction with references to phyllotaxis is provided at [1].

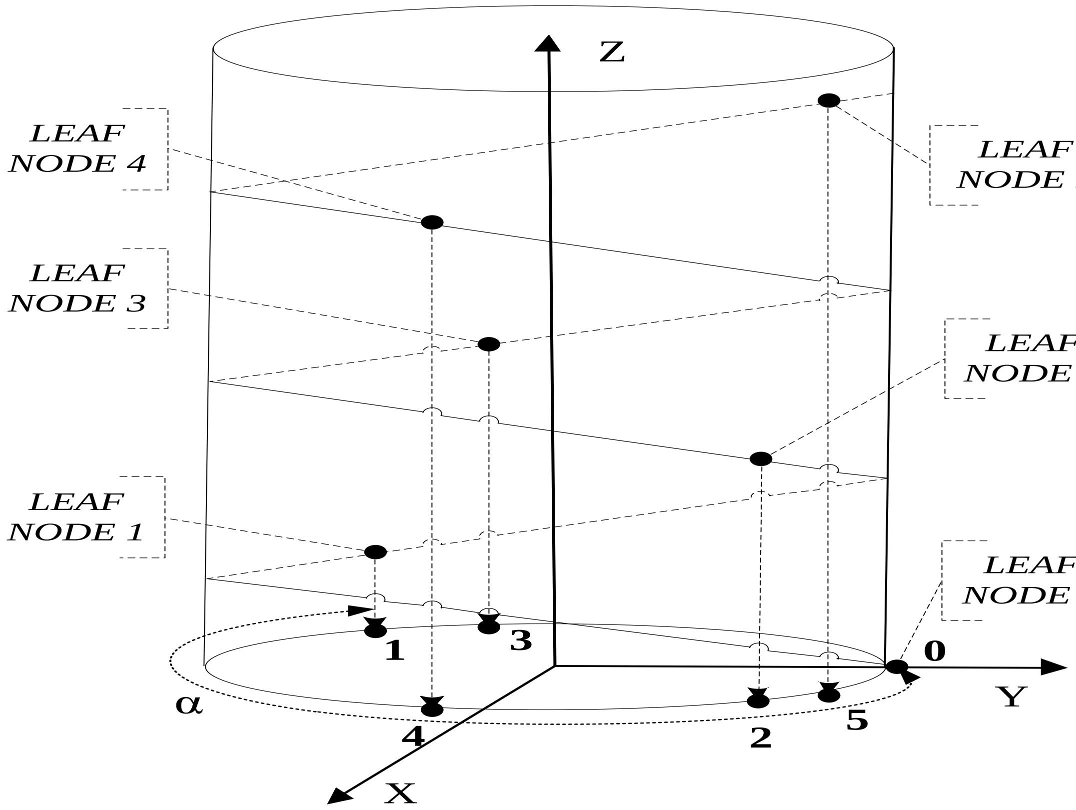

Figure 1 The emergence of leaf nodes on the stem of a plant

The spiral in Figure 1 may be represented in parametric form by $\boldsymbol{r}(t) = (1/2\pi)\cos(2\pi\alpha t)\boldsymbol{i} + (1/2\pi)\sin(2\pi\alpha t)\boldsymbol{j} + t\boldsymbol{k}$, for $t \geq 0$. If the parameter t represents time then the point on the unit circle labeled 0 has coordinates $(1/2\pi, 0, 0)$ and represents leaf node 0 which emerged at time $t = 0$. A point labeled n on the unit circle represents the projection of leaf node n at time $t = n$ which is at a point on the stem with coordinates $((1/2\pi)\cos(2\pi\alpha n), (1/2\pi)\sin(2\pi\alpha n), n)$. The distance along the spiral from leaf node n placed at time $t = n$ to leaf node $n + 1$ placed at time $t = n + 1$ is $\sqrt{(\alpha^2 + 1)}$.

2. Notation and basic relationships

Notation and basic relationships are presented for subsequent use throughout the article.

(a) The Fibonacci numbers $F_0 = 0$, $F_1 = 1 = F_2$, $F_3 = 2$, $F_4 = 3$, **…** satisfy the relationship

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2. \quad (1)$$

(b) The Golden Section $\tau = (\sqrt{5} - 1)/2 \approx 0.618$ and is the positive root of the quadratic equation

$$\tau^2 + \tau - 1 = 0. \quad (2)$$

(c) $\sqrt{5}F_n = \tau^n - (-1)^n\tau^{-n}$, which is referred to as the Binet expression. (3)

(d) $\tau^n = (-1)^n(F_{n-1} - \tau F_n)$ and $\tau^{-n} = F_{n+1} + \tau F_n$, for $n \geq 1$. (4)

(e) As $n \to \infty$, $F_n/F_{n+1} \to \tau$. (5)

(f) The real number $X = [X] + \{X\}$ where $[X]$ is the integer part of X (i.e. the largest integer not greater than X) and $0 \leq \{X\} < 1$ is the fractional part of X.

Also, $\{-X\} = 1 - \{X\}$. (6)

(g) For $n \geq 1$ and $1 \leq j \leq F_{n+1}$, $[(j + F_n)\tau] - [j\tau] - F_{n-1} = 0$ [2]. (7)

3. The golden placement policy

The golden placement policy (GPP) refers to the case where the placement of points on the unit circle uses $\alpha = \tau$ (the Golden Section) as defined in (2). Figure 2 illustrates the placement of the first 10 points following the GPP approach with labels on points represented by $u_1, u_2, u_3, \ldots, u_{10}$ in clockwise order around the circle starting from the point labeled $0 = u_1$.

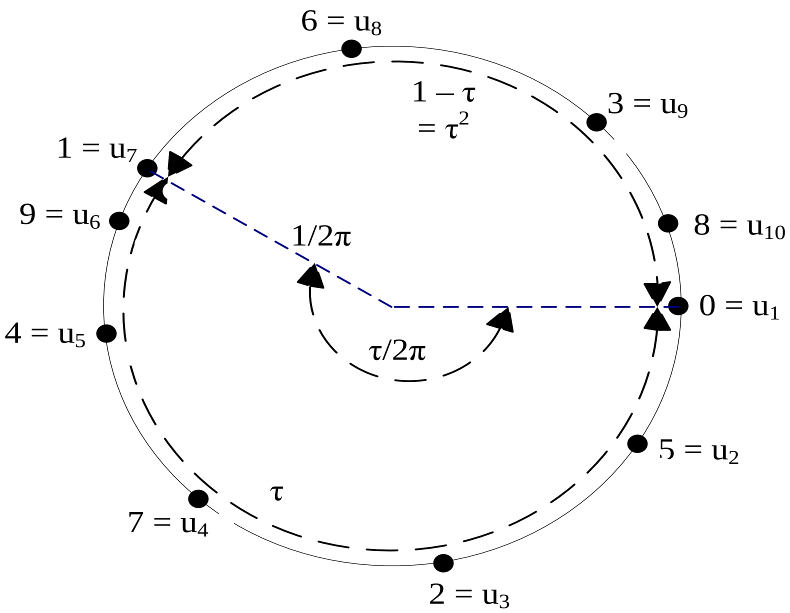

Figure 2 Ten points placed clockwise on the unit circle with $\alpha = \tau$

The motivation for using $\alpha = \tau$ is described in the following experiment where the placement of points is done in accordance with the policy of placing the next point in the oldest gap among all of the largest gaps:

At time t = 0 the point labeled 0 is placed on the unit circle and $1/2 < \alpha < 1$ (α irrational).

At time t = 1 the point labeled 1 is placed in a clockwise direction from the point labeled 0 at a distance α to form two gaps: 01 (length α, age 0) and 10 (length $1 - \alpha$, age 0) where 01 is the largest gap;

At time t = 2 the point labeled 2 is placed at a clockwise distance α from the point labeled 1 in the gap 01. At this time there are three gaps: gap 02 (length $2\alpha - 1$, age 0); gap 21 (length $1 - \alpha$, age 0); and gap 10 (length $1 - \alpha$, age 1). If α is restricted so that $1 - \alpha > 2\alpha - 1 > 0$ (i.e. $1/2 < \alpha < 2/3$) then the oldest gap among the largest gaps is the gap 10;

At time t = 3 the point labeled 3 is placed at a clockwise distance of α from the point labeled 2 in the gap 10 and now there are four gaps: 02 (length $2\alpha - 1$, age 1); 21 (length $1 - \alpha$, age 1); 13 (length $2\alpha - 1$, age 0); and 30 (length $2 - 3\alpha$, age 0). Because $1/2 < \alpha < 2/3$ the gap 21 is the oldest among the largest.

At time t = 4 the point labeled 4 is placed at a clockwise distance of α from the point labeled 3 in the gap 21 and now there are five gaps: 02 (length $2\alpha - 1$, age 2); 24 (length $2\alpha - 1$, age 0); 41 (length $2 - 3\alpha$, age 0); 13 (length $2\alpha - 1$, age 1); and 30 (length $2 - 3\alpha$, age 1). If α is restricted so that $2\alpha - 1 > 2 - 3\alpha > 0$ (i.e. $3/5 < \alpha < 2/3$) then the oldest gap among the largest gaps is the gap 02 and this is where the point labeled 5 will be placed at a clockwise distance of α from the point labeled 4.

Consequently, if this policy is continued then the pattern of upper and lower bounds on the values of α is: $1/2 < 3/5 < 8/13 < \ldots < \alpha < \ldots < 5/8 < 2/3 < 1$. It is noted from (1) that in terms of Fibonacci numbers the sequence of lower bounds on α is $F_2/F_3 < F_4/F_5 < F_6/F_7, \ldots$ and the sequence of upper bounds is $F_1/F_2 > F_3/F_4 > F_5/F_6, \ldots$ and so from (5) the use of this policy means that $\alpha = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \tau$.

The golden placement policy (GPP) is equivalent to a policy whereby the next point is placed in the oldest gap among all of the largest gaps. It is noted that this is not equivalent to placing the next point in the largest gap among all the oldest gaps and this is illustrated as part of the discussion following Table 1. When a new point is placed GPP focuses on the largest gaps and in particular the oldest of these as the one that will be split by the placement of the next point. In fact, as shown below the average length of a gap is 1/N, where N is the number of points that have been placed on the circle. It has been suggested that from the point of view of the emergence of leaf nodes on the stem of the plant that this allows the amount of shading on older leaves further down the stem due to new leaves further up the stem to be small and in that sense allowing a better sharing of access to light from above the stem. In terms of transmitters and receivers it leaves a reasonable distance between them and reduces overlap between them.

4. Patterns among Points and Gaps with the GPP

Table 1 illustrates patterns among points and gaps associated with the use of the GPP which were observed during the construction of Figure 2. As points are placed at successive intervals of time (t) the gaps formed between adjacent points are represented by u_iu_{i+1} where u_i is the label on the endpoint of the gap which is closer to the starting point labeled $0 = u_1$. At time t = N – 1 there are N points in place labeled 0, 1, 2, …, N – 1. The length of each gap is shown as well as its age. The gap selected for the placement of the next point is in bold type (i.e. the oldest gap among all of the largest gaps). Based on the length of the gaps at time t they are classified as small (S), medium (M), or large (L) and the number of each type of gap is shown.

Table 1 Patterns of points and gaps associated with using the GPP

Time t (Number of Points)	Characteristics of Gaps		Closest point to u_1 clockwise (counterclockwise)	Number of Gap Types	Number of Each Type of Gap
0 (1)	Gap	00	None	1	1L
	Length	1			
	Age	0			
	Type	L			
1 (2)	Gap	01 10	1	2	1S 1L
	Length	τ τ^2			
	Age	0 0			
	Type	L S			
2 (3)	Gap	02 21 10	2 (1)	2	1S 2L
	Length	τ^3 τ^2 τ^2			
	Age	0 0 1			
	Type	S L L			
3 (4)	Gap	02 21 13 30	2 (3)	3	1S 2M 1L
	Length	τ^3 τ^2 τ^3 τ^4			
	Age	1 1 0 0			
	Type	M L M S			
4 (5)	Gap	02 24 41 13 30	2 (3)	2	2S 3L
	Length	τ^3 τ^3 τ^4 τ^3 τ^4			
	Age	2 0 0 1 1			
	Type	L L S L S			
5 (6)	Gap	05 52 24 41 13 30	5 (3)	3	1S 3M 2L
	Length	τ^5 τ^4 τ^3 τ^4 τ^3 τ^4			
	Age	0 0 1 1 2 2			
	Type	S M L M L M			
6 (7)	Gap	05 52 24 41 16 63 30	5 (3)	3	2S 4M 1L
	Length	τ^5 τ^4 τ^3 τ^4 τ^5 τ^4 τ^4			
	Age	1 1 2 2 0 0 3			
	Type	S M L M S M M			
7 (8)	Gap	05 52 27 74 41 16 63 30	5 (3)	2	3S 5L
	Length	τ^5 τ^4 τ^5 τ^4 τ^4 τ^5 τ^4 τ^4			
	Age	2 2 0 0 3 1 1 4			
	Type	S L S L L S L L			
8 (9)	Gap	05 52 27 74 41 16 63 38 80	5 (8)	3	1S 4M 4L
	Length	τ^5 τ^4 τ^5 τ^4 τ^4 τ^5 τ^4 τ^5 τ^6			
	Age	3 3 1 1 4 2 2 0 0			
	Type	M L M L L M L M S			
9 (10)	Gap	05 52 27 74 49 91 16 63 38 80	5 (8)	3	2S 5M 3L
	Length	τ^5 τ^4 τ^5 τ^4 τ^5 τ^6 τ^5 τ^4 τ^5 τ^6			
	Age	4 4 2 2 0 0 3 3 1 1			
	Type	M L M L M S M L M S			

In Table 1 it is noted from (2) that when a point is placed in the gap u_iu_{i+1} the ratio of the lengths of the two newly formed gaps u_iu_{i+1} and $u_{i+1}u_{i+2}$ is $1: \tau = (1 + \tau) : 1$. In Table 2, it is seen that when the number of points in place is 7 that the oldest gap is 30 and it is of age 3 but this is not the gap that is selected for the placement of the next point labeled 7. Instead the gap 24 which is the single largest gap is selected and it is of age 2. This illustrates the result that the GPP approach selects the largest gap(s) and then the oldest among those. It does not select the oldest gap(s) and then the largest among those even though this approach would produce the same result as the GPP approach for all of the other cases illustrated in Table 1.

4.1 The order of the points on the circle

It is observed in Table 1 that the points which are closest to the point labeled 0 in a clockwise direction are always labeled with Fibonacci numbers (1, 2, 5, 13, …) and the same is true for the points closest to the point labeled 0 in a counter clockwise direction (1, 1, 3, 8, …). This follows from the following results:

(a) If the point labeled $N - 1$ has just been placed at time $t = N - 1$ (i.e. N points have been placed on the circle) and $F_{2n-1} \le N - 1 < F_{2n+1}$ then the closest point to the point labeled 0 in a clockwise direction is labeled F_{2n-1}.

(b) If the point labeled $N - 1$ has just been placed at time $t = N - 1$ and $F_{2n} \le N - 1 < F_{2n+2}$ then the closest point to the point labeled 0 in a counter clockwise direction is labeled F_{2n}.

(c) If the point labeled $N - 1$ has just been placed at time $t = N - 1$ and $F_n \le N - 1 < F_{n+1}$ then the closest point to the point labeled 0: **(i)** In a counter clockwise direction from 0 is labeled F_n if n is even; and **(ii)** In a clockwise direction from 0 is labeled F_n if n is odd.

Proofs

The distance to the point labeled $N - 1$ is $\{(N-1)\tau\}$ in a clockwise direction from the point labeled 0 and $1 - \{(N-1)\tau\} = \{-(N-1)\tau\}$ in a counter clockwise direction.

(a) From (4), $\tau F_{2n-3} = \tau^{2n-3} + F_{2n-4}$ and $\tau F_{2n-1} = \tau^{2n-1} + F_{2n-2}$ and so $\{\tau F_{2n-1}\}/\{\tau F_{2n-3}\} = \tau^{2n-1}/\tau^{2n-3} = \tau^2 < 1$. Consequently, the point labeled F_{2n-1} is closer in a clockwise direction to the point labeled 0 than the point labeled F_{2n-3}, for $n \ge 2$.

From (7), for $n \ge 1$ and $1 \le j < F_{2n}$, $[(F_{2n-1} + j)\tau] - [j\tau] - F_{2n-2} = 0 < \{j\tau\}$ and so $[(F_{2n-1} + j)\tau] - F_{2n-2} < j\tau$. Thus,

$$(F_{2n-1} + j)\tau - \{(F_{2n-1} + j)\tau\} - F_{2n-2} < j\tau. \quad (8)$$

From (4), $\tau F_{2n-1} = F_{2n-2} + \tau^{2n-1}$ and so $F_{2n-2} = [\tau F_{2n-1}] = \tau F_{2n-1} - \{\tau F_{2n-1}\}$ substituted in (8) gives $(F_{2n-1} + j)\tau - \{(F_{2n-1} + j)\tau\} - \tau F_{2n-1} + \{\tau F_{2n-1}\} < j\tau$ and so $\{\tau F_{2n-1}\} < \{(F_{2n-1} + j)\tau\}$ for $j = 1, 2, 3, \ldots, F_{2n} - 1$ ■

(b) From (4), $\tau F_{2n-2} + \tau^{2n-2} = F_{2n-3}$ and so $\{\tau F_{2n-2}\} = 1 - \tau^{2n-2}$ and similarly $\{\tau F_{2n}\} = 1 - \tau^{2n}$ with the result that $\{\tau F_{2n-2}\} - \{\tau F_{2n}\} = \tau^{2n} - \tau^{2n-2} < 0$. Consequently, the point labeled F_{2n-2} is closer in a clockwise direction to the point labeled 0 than the point labeled F_{2n}, for $n \ge 2$.

From (7), it follows that,

$$(F_{2n} + j)\tau - \{(F_{2n} + j)\tau\} - j\tau + \{j\tau\} - F_{2n-1} = 0. \quad (9)$$

From (4), it follows that $\tau F_{2n} - \{\tau F_{2n}\} = F_{2n-1} - 1$, which substituted in (9) gives $\{\tau F_{2n}\} - \{(F_{2n} + j)\tau\} = 1 - \{j\tau\} > 0$ and so $\{\tau F_{2n}\} > \{(F_{2n} + j)\tau\}$ for $j = 1, 2, 3, \ldots, F_{2n+1} - 1$ ■

(c) From **(a)** and **(b)** it follows that if:

(i) n is even then the first point in a clockwise direction from the point labeled 0 is labeled F_{n-1} and the first point in a counterclockwise direction for the point labeled 0 is labeled F_n. It follows from (4) that $\{\tau F_{n-1}\} = \tau^{n-1}$ and $\{\tau F_n\} = 1 - \tau^n$. The result follows from $1 - \{\tau F_n\} = \tau^n < \tau^{n-1} = \{\tau F_{n-1}\}$.

(ii) n is odd then the first point in a clockwise direction from the point labeled 0 is labeled F_n and the first point in a counter clockwise direction for the point labeled 0 is labeled F_{n-1}. It follows from (4) that $\{\tau F_n\} = \tau^n$ and $\{\tau F_{n-1}\} = 1 - \tau^{n-1}$. The result follows from $1 - \{\tau F_{n-1}\} = \tau^{n-1} < \tau^n = \{\tau F_n\}$ ■

As illustrated in Figure 2 the sequence (u_j, $j = 1, 2, 3, \ldots, N$) represents the labels on the N points on the circle ordered in a clockwise direction starting with $u_1 = 0$. The results above determine the values of u_2 and u_N so that if $F_n \le N - 1 < F_{n+1}$,

$$u_2 = \begin{cases} F_{n-1}, n \text{ even, and} \\ F_n, n \text{ odd,} \end{cases} \qquad u_N = \begin{cases} F_n, n \text{ even,} \\ F_{n-1}, n \text{ odd.} \end{cases} \tag{10}$$

[3], [4], and [have shown that for j = 1, 2, 3, …, N – 1,

$$u_{j+1} - u_j = \begin{cases} u_2, & 0 \le u_j < N - u_2, \\ u_2 - u_N, & N - u_2 \le u_j < u_N, \\ -u_N, & u_N \le u_j < N. \end{cases} \tag{11}$$

If $F_n \le N - 1 < F_{n+1}$ then substituting (10) in (11) gives, for j = 2, 3, 4, …, N – 1,

$$u_{j+1} - u_j = \begin{cases} F_{n-1}, & 0 \le u_j < N - F_{n-1}, & \\ -F_{n-2}, & N - F_{n-1} \le u_j < F_n, & \text{for } n \text{ even,} \\ -F_n, & F_n \le u_j < N. & \\ F_n, & 0 \le u_j < N - F_n, & \\ F_{n-2}, & N - F_n \le u_j < F_{n-1}, & \text{for } n \text{ odd.} \\ -F_{n-1}, & F_{n-1} \le u_j < N, & \end{cases} \tag{12}$$

From (12) if the number of points $N = F_{n+1}$ then for j = 2, 3, 4, …, N – 1,

$$u_{j+1} - u_j = \begin{cases} F_{n-1}, & 0 \le u_j < F_n, & \text{for } n \text{ even,} \\ -F_n, & F_n \le u_j < F_{n+1}, & \\ F_n, & 0 \le u_j < F_{n-1}, & \text{for } n \text{ odd.} \\ -F_{n-1}, & F_{n-1} \le u_j < F_{n+1}, & \end{cases} \tag{13}$$

4.2 The lengths, ages, and types of gaps

From (10) using (4) if $F_n \le N - 1 < F_{n+1}$ the length of the first gap $0u_2$ in a clockwise direction is $\{\tau F_{n-1}\} = \tau^{n-1}$ for n even and $\{\tau F_n\} = \tau^n$ for n odd. Similarly, the length of the first gap $0u_N$ in a counter clockwise direction is $1 - \{\tau F_n\} = \tau^n$ for n even and $1 - \{\tau F_n \tau\} = \tau^{n-1}$ for n odd.

The lengths of the gaps around the circle in a clockwise direction from the point labeled 0 are determined from (12). If $F_n \le N - 1 < F_{n+1}$ then using (6) for j = 2, 3, 4, …, N – 1, the length of the gap $u_j u_{j+1}$ is given by,

$$\{(u_{j+1} - u_j)\tau\} = \begin{cases} \{\tau F_{n-1}\} = & \tau^{n-1}, & 0 \le u_j < N - F_{n-1}, & \\ \{-\tau F_{n-2}\} = & \tau^{n-2}, & N - F_{n-1} \le u_j < F_n, & \text{for } n \text{ even,} \\ \{-\tau F_n\} = & \tau^n, & F_n \le u_j < N. & \\ \{\tau F_n\} = & \tau^n, & 0 \le u_j < N - F_n, & \\ \{\tau F_{n-2}\} = & \tau^{n-2}, & N - F_n \le u_j < F_{n-1}, & \text{for } n \text{ odd.} \\ \{-\tau F_{n-1}\} = & \tau^{n-1}, & F_{n-1} \le u_j < N, & \end{cases} \tag{14}$$

In the particular case where the number of points N is the Fibonacci number F_{n+1} then from (13) or (14) for j = 2, 3, 4, …, N – 1,

$$\{(u_{j+1} - u_j)\tau\} = \begin{cases} \{\tau F_{n-1}\} = & \tau^{n-1}, & 0 \le u_j < N - F_{n-1}, & \text{for } n \text{ even,} \\ \{-\tau F_n\} = & \tau^n, & F_n \le u_j < F_{n+1}, & \\ \{\tau F_n\} = & \tau^n, & 0 \le u_j < F_{n-1}, & \text{for } n \text{ odd.} \\ \{-\tau F_{n-1} = & \tau^{n-1}, & F_{n-1} \le u_j < F_{n+1}, & \end{cases} \tag{15}$$

From (14) it is seen that if $F_n < N < F_{n+1}$ (i.e. N is not the Fibonacci number F_{n+1}) then there are three different lengths for the gaps τ^n, τ^{n-1}, and τ^{n-2} which are described as small (S), medium (M), and large (L), respectively. However, from (15) it is seen that if $N = F_{n+1}$ then there are only two different lengths for the gaps given by τ^n (S) and τ^{n-1} (L). As reported in [6], Steinhaus was the first to conjecture this result, which does not hold for all values of the distance α, and is often referred to as the Three Gap Theorem or the Steinhaus conjecture.

Based on the findings reported so far it is possible to describe other patterns and properties among the lengths, ages, and types of gaps. These are presented in Table 2 where the number of points N is not a Fibonacci number and in Table 3 where the number of points is a Fibonacci number.

Table 2 Patterns among lengths, ages, and types of gaps when the number of points is not a Fibonacci number.

Number of Points $N = F_n + k$, for $k = 1, 2, 3, \ldots, F_{n-1} - 1$	Type of Gap		
	Large	Medium	Small
Gaps and their Age	For $m = k, k + 1, k + 2, \ldots, F_{n-1} - 1$, Gap $u_j u_{j+1}$ $u_j = F_{n-2} + m$, $u_{j+1} = m$, n even, $u_j = m$, $u_{j+1} = F_{n-2} + m$, n odd. Age of Gap $u_j u_{j+1}$ is $F_{n-1} - 1 - m + k$	For $m = 0, 1, 2, \ldots, F_{n-2} + k - 1$, Gap $u_j u_{j+1}$ $u_j = m$, $u_{j+1} = F_{n-1} + m$, n even, $u_j = F_{n-1} + m$, $u_{j+1} = m$, n odd. Age of Gap $u_j u_{j+1}$ is $F_{n-2} - 1 - m + k$	For $m = 0, 1, 2, \ldots, k - 1$, Gap $u_j u_{j+1}$ $u_j = F_n + m$, $u_{j+1} = m$, n even, $u_j = m$, $u_{j+1} = F_n + m$, n odd. Age of Gap $u_j u_{j+1}$ is $k - 1 - m$
Lengths of Gaps	τ^{n-2}	τ^{n-1}	τ^n
Number of Gaps	$F_{n-1} - k$	$F_{n-2} + k$	k
Total length of Gaps	$(F_{n-1} - k)\,\tau^{n-2}$	$(F_{n-2} + k)\,\tau^{n-1}$	$k\,\tau^n$
Oldest Gap	$F_{n-2} + kk$, n even, $kF_{n-2} + k$, n odd.	$0F_{n-1}$, n even, $F_{n-1}0$, n odd.	F_n0, n even, $0F_n$, n odd.
Age of Oldest Gap	$F_{n-1} - 1$	$F_{n-2} + k - 1$	$k - 1$
Youngest Gap	$F_n - 1F_{n-1} - 1$, n even, $F_{n-1} - 1F_n - 1$, n odd.	$F_{n-2} + k - 1F_n + k - 1$, n even, $F_n + k - 1F_{n-2} + k - 1$, n odd.	$F_n + k - 1k - 1$, n even, $k - 1F_n + k - 1$, n odd.
Age of Youngest Gap	k	0	0
Mean Age of Gaps	$(F_{n-1} - 1 + k)/2$	$(F_{n-2} - 1 + k)/2$	$(k - 1)/2$

From the total length of all of the gaps shown Table 2 it is seen that:

(a) Because the circle has unit length then for $n \geq 1$ and $k = 1. 2, 3, \ldots, F_{n-1} - 1$, $1 = (F_{n-1} - k)\,\tau^{n-2} + (F_{n-2} + k)\,\tau^{n-1} + k\tau^n = \tau^n((F_{n-1} - k)\tau^{-2} + (F_{n-2} + k)\tau^{-1} + k) = \tau^n(F_{n+1} + \tau F_n)$ which verifies (4). Also, the average length of the gaps is $1/(F_n + k) = 1/N$.

(b) The length of a large gap **:** the length of a medium gap **:** length of a small gap is $\tau^{n-2} : \tau^{n-1} : \tau^n = \tau + 2 : \tau + 1 : \tau$.

(c) If there are exactly $N = F_n + k$, ($k = 1, 2, 3, \ldots, F_{n-1} - 1$) points on the circle then when the next point labeled N is placed clockwise in the oldest of all of the largest gaps, which is $F_{n-2} + kk$ (n even) and $kF_{n-2} + k$ (n odd), the ratio of the lengths of the newly formed gaps is $1 : \tau$ (n even) and $\tau : 1$ (n odd).

Proof

A proof for **(c)** is provided for the case where n is even. The proof is similar for the case where n is odd.

If $N = F_n + k$, ($k = 1, 2, 3, \ldots, F_{n-1} - 1$) (n even) then the largest of the oldest gaps is $F_{n-2} + kk$ and the next point will be placed in this gap and will have the label $F_n + k$. Consequently, the ratio of the lengths of the newly formed gaps will be $\{(F_n + k - (F_{n-2} + k))\tau\} : \{(k - (F_n + k))\tau\} = \{\tau F_{n-1}\} : \{-\tau F_n\} = \tau^{n-1} : 1 - \{\tau F_n\} = \tau^{n-1} : \tau^n = \tau^{-1} : 1 = 1 : \tau$ ■

In the particular case where the number of points N is the Fibonacci number F_n then patterns and properties among the lengths, ages, and types of gaps are summarized in Table 3.

Table 3 Patterns among lengths, ages, and types of gaps when the number of points is a Fibonacci number

Number of Points	Type of Gap	
$N = F_n$	Large	Small
Gaps and their Age	For $m = 0, 1, 2, \ldots, F_{n-1} - 1$, Gap u_ju_{j+1}: $u_j = F_{n-2} + m$, $u_{j+1} = m$, n even, $u_j = m$, $u_{j+1} = F_{n-2} + m$, n odd. Age of Gap u_ju_{j+1} is $F_{n-1} - 1 - m$	For $m = 0, 1, 2, \ldots, F_{n-2} - 1$, Gap u_ju_{j+1}: $u_j = m$, $u_{j+1} = F_{n-1} + m$, n even, $u_j = F_{n-1} + m$, $u_{j+1} = m$, n odd. Age of Gap u_ju_{j+1} is $F_{n-2} - 1 - m$
Length of Gaps	τ^{n-2}	τ^{n-1}
Number of Gaps	F_{n-1}	F_{n-2}
Total length of Gaps	$F_{n-1}\tau^{n-2}$	$F_{n-2}\tau^{n-1}$
Oldest Gap	$F_{n-2}0$, n even, $0F_{n-2}$, n odd.	$0F_{n-1}$, n even, $F_{n-1}0$, n odd.
Age of Oldest Gap	$F_{n-1} - 1$	$F_{n-2} - 1$
Youngest Gap	$F_n - 1F_{n-1} - 1$, n even, $F_{n-1} - 1F_n - 1$, n odd.	$F_{n-2} - 1F_n - 1$, n even, $F_n - 1F_{n-2} - 1$, n odd.
Age of Youngest Gap	0	0
Mean Age of Gaps	$(F_{n-1} - 1)/2$	$(F_{n-2} - 1)/2$
Variance of the Age of Gaps	$(F_{n-1} - 1)(F_{n-1} + 1)/12$	$(F_{n-2} - 1)(F_{n-2} + 1)/12$

From Table 3 it is seen that if $N = F_n$ then:

(a) The mean length of gaps is $1/F_n = 1/N$ with variance $\frac{F_{n-1}}{F_n}\left(1-\frac{F_{n-1}}{F_n}\right)\tau^{2n}$ which is approximately τ^{2n+3} for large values of $N = F_n$;

(b) The mean age of all the gaps is $\frac{F_{n-1}^2 + F_{n-2}^2 - F_n}{2F_n}$;

(c) The ratio of the mean age of small gaps to the mean age of large gaps approaches τ for large values of $N = F_n$ and the corresponding ratio of the variances approaches $\tau^2 : 1 = 1 - \tau : 1$ for large values of F_n;

(d) Among the oldest gaps the age of a small gap**:** the age of a large gap $= F_{n-2} - 1 : F_{n-1} - 1$ which is approximately $\tau : 1$ for large values of F_n.

(e) If there are exactly F_n points on the circle then when the next point labeled N is placed clockwise in the oldest of all of the largest gaps, which is $F_{n-2}\,0$ for n even and $0F_{n-2}$ for n odd, the ratio of the lengths of the newly formed gaps is $1 : \tau$ for n even and $\tau : 1$ for n odd. The proofs follow in the same manner as in **(c)** above for the case where $N = F_n + k$, $(k = 1, 2, 3, \ldots, F_{n-1} - 1)$.

5. Conclusion

The purpose of this article was to describe patterns and properties among the points and the gaps that are formed when points are placed successively in a clockwise direction around a unit circle so that the distance between successive placements is the Golden Section (τ). It is shown that this situation described as the golden placement policy (GPP) is equivalent to a policy whereby the next point is placed clockwise in the oldest gap among all of the largest gaps. The analyses described the ordering of the points; the length, age, type, and number of gaps formed; and revealed the importance of Fibonacci numbers in describing the patterns and properties that emerged. The analyses provide insights into practical situations including the placement of transmitters and receivers and phyllotaxis.

The examination of the sequences of gap types around the circle involving S, M, and L was not considered in detail in this article. However, investigating these sequences of gap types leads to numerous other interesting patterns and properties which are studied in detail in [3], [7], and [8]. It is strongly recommended to the interested reader that they follow up this article by consulting these references.

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