KKU Res. J. 2013; 18(2): 344-349
http : //resjournal.kku.ac.th

ไบไอดีลพีปรกติของเนียร์เลฟท์ออลโมซท์ริง
Bi-ideals of a P-Regular Near Left Almost Rings

*ธิติ เกตุคำ**
Thtii Gaketem

คณะวิทยาศาสตร์ มหาวิทยาลัยพะเยา พะเยา 56000
**Correspondent author: newtonisaac41@yahoo.com*

บทคัดย่อ

ในงานวิจัยชิ้นนี้เราจะนิยามและศึกษาสมบัติของไบไอดีลบนพีปรกติของเอ็นแอลเอริง

Abstract

In this paper, we define and study properties of bi-ideals of P- regular nLA-ring

คำสำคัญ : เอ็นแอลเอริง, พีปรกติเอ็นแอลเอริง, ไบไอดีล
Keywords : nLA-ring, P-regular nLA-ring, Bi-ideal

1. Introduction

Kazim and Naseeruddin (1)[Defintion 2.1] have introduced a pseudo associate law or left invertive law in a groupoid G by

$(ab)c = (cb)a$ for all $a, b, c \in G$

and have named it as left invertive law, and called the groupoid a left almost semigroup (abbreviated as an LA-semigroup) if it satisfies left invertive law. The groupoid G is also called an Abel-Grassmann's groupoids (abbreviated as an AG-groupoids), see (1) or (2). It is a nonassociative algebraic structure midway between a groupoid and a commutative semigroup.

Kazim and Naseeruddin (1) [Proposition 2.1] asserted that, in every LA-semigroups G a medial law hold

$(ab)(cd) = (ac)(cd)$ for all $a, b, c, d \in G$

Mushtab and Khan (3) asserted that, in every LA-semigroups G with left identity

$(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in G$

Further Khan, Faisal, and Amjid (2), asserted that, if a LA-semigroup G with left identity the following law holds

$a(bc) = b(ac)$ for all $a, b, c \in G$

Sarwar (Kamran) (4) defined LA-group as the following; a groupoid G is called a left almost group, abbreviated as LA-group, if

(1) there exists $e \in G$ such that $ea = a$ for all $a \in G$,

(2) for every $a \in G$ there exists $a^{-1} \in G$ such that, $a^{-1}a = e$

(3) $(ab)c = (cb)a$ for all $a, b, c \in G$.

Yusuf introduces the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations "+" and "×" is called a left almost ring, if $(R, +)$ is an LA-group, $(R,.)$ is an LA-semigroup and distributive laws of "•" over "+" holds. Shah and Rehman (5) asserted that a commutative ring $(R, +, .)$ we can always obtain an LA-ring $(R, \oplus, .)$ by defining, for $a, b \in R$, $a \oplus b = b - a$ and ab is same as in the ring. We cannot assume the addition to be commutative in an LA-ring. An LA-ring $(R, +, \bullet)$ is said to be LA-integral domain if $ab = 0$ for all $a, b \in R$ then $a = 0$ or $b = 0$.

Let $(R, +, \bullet)$ be an LA-ring and S be a non-empty subset of R and S is itself and LA-ring under the binary operation induced by R, the S is called an LA-subring of R, then S is called an LA-subring of

$(R, +, .)$. If S is an LA-subring of an LA-ring $(R, +, .)$

then S is called a left ideal of R if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.

By (6) a near-ring is a non-empty set N together with two binary operations "+" and "•" such that $(N, +)$ is a group (not necessarily abelian), $(N, \bullet)$ is a semigroup and one sided distributive (left or right) of "•"over "+" holds.

By (7) If a subgroup B of $(N, +)$ has the property $BNB \subseteq B$, then it is called a bi-ideal of N.

By (8) a near-ring N is a regular if for each $x \in N$; there exists $y \in N$ such that $xyx = x$. A regular near-ring was introduced by J.C. Beidlemann in 1968 and later S. Leigh and H.E. Heatherly etc. studied the structure of a regular near-ring. Let N be a near-ring with the unity and P be an ideal of N. Then the near-ring N is said to be a P-regular near-ring if for each $a \in N$; there exists $x \in N$ such that $axa - a \in P$. If $P = 0$, then a P-regular near-ring is a regular near-ring. Hence the notion of P-regularity is a generalization of regularity.

2. Near Left Almost Rings

T. Shah, F. Rehman and M. Raees (5) introduces the concept of a near left almost ring (nLA-ring).

Definition 2.1. (9) A non-empty set N with two binary operation "+" and "•" is called a near left almost ring (or simply an nLA-ring) if and only if

(1) $(N, +)$ is an LA-group.

(2) $(N, \bullet)$ is an LA-semigroup.

(3) Left distributive property of • over + holds, that is $a(b+c) = ab+ ac$ for all $a, b, c \in N$.

Definition 2.2. (9) An nLA-ring $(N, +, \bullet)$ with left identity 1; such that $1a = a$ for all $a \in N$, is called an nLA-ring with left identity.

Definition 2.3. (9) A nonempty subset S of an nLA-ring N is said to be an nLA-subring if and only if S is itself an nLA-ring under the same binary operations as in N.

Theorem 2.4. (9) A non-empty subset S of an nLA-ring $(N, +, \bullet)$ is an nLAsubring if and only if $a - b \in S$ and $ab \in S$ for all $a, b \in S$.

Definition 2.5. (9) An nLA-subring I of an nLA-ring N is called a left ideal of N if $NI \subseteq I$, and I is called a right ideal if for all $n, m \in N$ and $i \in I$ such that $(i + n)m - nm \in I$; and is called two sided ideal or simply ideal if it is both left and right ideal.

3. Bi-ideals of a P-Regular Near Left Almost Rings

Next we defines of a regular, bi-ideal and P-regular in nLA-ring is defines the same as a regular, bi-ideal and P-regular in near-ring in (10).

Definition 3.1. A nLA-ring N is called a regular nLA-ring if for each $x \in N$ there exists $y \in N$ such that $xyx = x$.

Definition 3.2. If a LA-subgroup B of $(N, +)$ has the property $BN \cap NB \subseteq B$, then it is called a bi-ideal of N.

Lemma 3.3. Let N be an nLA-ring and B_1, B_2 are bi-ideal of N. Then $B_1 \cap B_2$ is a Bi-ideal of N.

Proof. Since B_1, B_2 are LA-subgroup of $(N, +)$ we have $B_1 \cap B_2$ is a LA-subgroup of $(N, +)$. We must show that $(B_1 \cap B_2)N(B_1 \cap B_2) \subseteq B_1 \cap B_2$. Then

$$\begin{aligned}(B_1 \cap B_2)N(B_1 \cap B_2) &\subseteq (B_1 \cap B_2)NB_1 \cap (B_1 \cap B_2)NB_2 \\ &\subseteq B_1N \cap B_2N \\ &\subseteq B_1 \cap B_2.\end{aligned}$$

Thus $B_1 \cap B_2$ is a Bi-ideal of N.

Definition 3.4. Let N be an nLA-ring with the unity and P be an ideal of N. Then the nLA-ring N is said to be a P-regular nLA-ring if for each $a \in N$, there exists $x \in N$ such that $axa - a \in P$.

Theorem 3.5. Let N be a P-regular nLA-ring. If $P = 0$, then a P-regular nLA-ring is a regular nLA-ring.

Proof. Let N be a P-regular nLA-ring, then for each $n \in N$, there exists $x \in N$ such that $nxn - n \in P$ that is $nxn - n = P$, where P is ideal of N. If $P = 0$ then $nxn - n = 0$ implies that $nxn = n$. Thus P is a regular nLA-ring.

The following theorems with proved is analogous as in (10)

Theorem 3.6. Let N be a P-regular nLA-ring. Then for each $n \in N$, there exists $n' \in N$ such that $n'n \in P$.

Proof. Let N be a P-regular nLA-ring, then for each $n \in N$, there exists $x \in N$ such that $nxn - n \in P$, where P is ideal of N. So $(nx - 1)n \in P$ and then put $n' = nx - 1$. Thus we obtain $n'n \in P$.

Theorem 3.7. Let N be a P-regular nLA-ring. Then for every left ideal L and every right ideal R of N, $(P + L) \cap (P + R) = P + LR$.

Proof. Suppose that N is a P-regular nLA-ring, L is a left ideal and R is a right ideal of N. If $n \in (P + L) \cap (P + R)$ then element n can be written as $n = p_1 + l$ and $n = p_2 + r$ for some $p_1; p_2 \in P; l \in L$ and $r \in R$. By definition P-regularity of N, $nxn - n \in P$ for some $x \in N$, which means that the element n can also be expressed in the form $n = -p + nxn$ for some $p \in P$. From these one the obtains.

$$\begin{aligned} n &= -p + nxn = -p + (p_1 + l)x(p_2 + r) \\ &= -p + (p_1 + l)(xp_2 + xr) \quad \text{; by Definition 2.1.(3)} \\ &= -p + [(p_1 + l)xp_2] + [(p_1 + l)xr] \quad \text{; by Definition 2.1.(3)} \\ &= -p + [(p_1 + l)xp_2] + [(p_1 + l)xr - lxr] + lxr \\ &= -p + [(p_1 + l)xp_2] + p_3 + lxr \; ; \; p_3 = [(p_1 + l)xr - lxr] \in P \\ &= p_4 + lxr \;\hat{\text{I}}\; P + LR \; ; \; p_4 = -p + [(p_1 + l)xp2] + p_3] \in P. \end{aligned}$$

Hence $(P + L) \cap (P + R) \subseteq P + LR$.

For the converse, if $n \in P + LR$, then the element n can be written as $n = p + lr$ for some $p \in P$, $l \in L$ and $r \in R$. Since L is a left ideal and R is a right ideal of N, it is obvious that $n = p + lr$ belongs to $(P + L) \cap (P + R)$ thus $P + LR \subseteq (P + L) \cap (P + R)$. Hence $(P + L) \cap (P + R) = P + LR$.

Now the question to be raised is what relationship is between a bi-ideal and the ideal P of a P-regular nLA-ring. It leads at once to the representation of elements of bi-ideals of a P-regular nLA-ring in connection with the ideal P. So the coming theorems present several representations of elements of bi-ideals of a P-regular nLA-ring.

Theorem 3.8. If N is a P-regular nLA-ring, then every element of a bi-ideal B of N can be represented as the sum of two elements of P and B.

Proof. Let N be a P-regular nLA-ring and B be a Bi-ideal of N. If $b \in B$, then there exists $x \in N$ such that $bxb - b \in P$, where P is ideal of N. So it has a representation $b = -p + bxb$ for some $p \in P$. By definition of a bi-ideal, $bxb \in BNB \subseteq B$ and therefore, we obtain $b = -p + bxb \in P + B$

Theorem 3.9. Let N be a P-regular nLA-ring, B_1 and B_2 are bi-ideals of N. If $b \in B_1 \cap B_2$, then the element p can be represented as

$$b = p + b_1xb_2$$

for some $p \in P$, $x \in N$, $b_1 \in B_1$ and $b_2 \in B_2$.

Proof. Suppose that N be a P-regular nLA-ring, B_1 and B_2 are bi-ideals of N. Then for each $b \in N$ there is $x \in N$. such that $bxb - b \in P$, where P is ideal of N.

If $b \in B_1 \cap B_2$ then by lemma 3.3 thus $B_1 \cap B_2$ is a Bi-ideal of N. By theorem 3.8, the element b of $B_1 \cap B_2$ we can be written as both

$b = p_1 + b_1$ and $b = p_2 + b_2$ for some $p_1, p_2 \in P$, $b_1 \in B_1$ and $b_2 \in B_2$. By P-regularity of N, the element b also has the form

$b = -p_3 + bxb$ for some $p_3 \in P$ and then it follows that

$$\begin{aligned}
b &= -p_3 + bxb = -p_3 + (p_1 + b_1)x(p_2 + b_2) \\
&= -p_3 + (p_1 + b_1)(xp_2 + xb_2) \text{ ; by Definition 2.1.(3)} \\
&= -p_3 + [(p_1 + b_1)xp_2] + [(p_1 + b_1)xb_2] \text{ ; by Definition 2.1.(3)} \\
&= -p_3 + [(p_1 + b_1)xp_2] + [(p_1 + b_1)xb_2 - b_1xb_2] + b_1xb_2 \\
&= -p_3 + [(p_1 + b_1)xp_2] + p4 + b_1xb_2 \text{ ; } p_4 = (p_1 + b_1)xb_2 - b1xb_2 \in P \\
&= p + b_1xb_2 \text{ ; } p = -p_3 + [(p_1 + b_1)xp_2] + p_4 \in P.
\end{aligned}$$

Hence $b = p + b_1xb_2$

Next, by induction, Theorem 3.9 can be extended to the case of the intersection of finitely many bi-ideals of a P-regular nLA-ring as follows.

Theorem 3.10. Let N be a P-regular nLA-ring, B_i be a bi-ideals of N for $1 \le i \le n$. If $B \in \cap^n_{i=1} B_i$ then the element B can be represented as

$$b = p + b_1xb_2xb_3x \cdots xb_{n-1}xb_n$$

for some $p \in P$, $x \in N$, $b_i \in B_i$, $i = 1, 2, \ldots, n$.

Proof. Let N be a P-regular nLA-ring, B_i be a bi-ideals of N for $1 \le i \le n$. By induction on i; if $b_i \in B_i$ then by Theorem 3.9, the element b can be represented as

$b = p + b_1$ thus $b_1 = - p + b$ for some $p \in P$ and $b_1 \in B_1$.

Assume that an element b of $\cap^n_{i=1} B_i$ can be represented as

$$b = p_1 + b_1xb_2xb_3x \cdots xb_{n-2}xb_{n-1}$$

for some $p_1 \in P$, $x \in N$, $b \in Bi$ $(1 \le i \le n\text{-}1)$.

If $b \in \cap^n_{i=1} B_i$, then by the trivial inclusion $\cap^n_{i=1} B_i \subseteq \cap^{n-1}_{i=1} B_i$ and the inductive assumption, the element b can be represented as $b = p + b_1xb_2xb_3x \cdots xb_{n-1}$ for some $p_1 \in P$; $x \in N$ and $b_i \in B_i (1 \le i \le n\text{-}1)$ by Theorem 3.8, it also has a representation $b = p_2 + b_n$ for some $p_2 \in P$ and $b_n \in B_n$.

Hence $bxb = (p + b_1xb_2xb_3x \cdots xb_{n-1})x(p_2 + b_n)$ and by P-regularity of N, the element b of $\cap^{n-1}_{i=1} B_i$ has another representation $b = - p_3 + bxb$ for some $p_3 \in P$ and $x \in N$. So we have the following,

$$\begin{aligned}
b &= - p_3 + bxb = - p_3 + (p + b_1xb_2xb_3x \cdots xb_{n-1})x(p_2 + b_n) \\
&= - p_3 + (p + b_1xb_2xb_3x \cdots xb_{n-1})(xp_2 + xb_n) \text{; by Definition 2.1.(3)} \\
&= - p_3 + (p + b_1xb_2xb_3x \cdots xb_{n-1})xp2 + (p + b_1xb_2xb_3x \cdots xb_{n-1})xbn \text{; by Definition 2.1.(3)} \\
&= - p_3 + (p + b_1xb_2xb_3x \cdots xb_{n-1})xp2 + (p + b_1xb_2xb_3x \cdots xb_{n-1})xbn \\
&\quad - (p + b_1xb_2xb_3x \cdots xb_{n-1}) + (p + b_1xb_2xb_3x \cdots xb_{n-1}) \\
&= - p_3 + (p + b_1xb_2xb_3x \cdots xb_{n-1})xp_2 + [(p + b_1xb_2xb_3x \cdots xb_{n-1})xb_n - (p + b_1xb_2xb_3x \cdots xb_{n-1})] + (p + b_1xb_2xb_3x \cdots xb_{n-1}) \\
&= p + (p + b_1xb_2xb_3x \cdots xb_{n-1}),
\end{aligned}$$

where $p = - p_3 + (p + b_1xb_2xb_3x \cdots xb_{n-1})xp_2 + [(p + b_1xb_2xb_3x \cdots xb_{n-1})xb_n - (p + b_1xb_2xb_3x \cdots xb_{n-1})] \in P$.

Hence $b = p + b_1xb_2xb_3x \cdots xb_{n-1}xb_n$, for $p \in P$.

Theorem 3.11. If N is a P-regular nLA-ring, then every bi-ideal B of N has the form

$$P + B = P + BNB$$

Proof. Assume that N is a P-regular nLA-ring and let B be a bi-ideal of N. by lemma 2.6 then $BNB \in B$ holds and it leads to $P + BNB \subseteq P + B$.

For the opposite direction, let $n \in P + B$, then the element n can be expressed as $n = p' + b'$ for some $p' \in P$ and $b' \in B$. By P-regularity of N, then there exists $x \in N$ such that $b'xb' - b' \in P$. Thus $b'xb' - b' = p''$ for some $p'' \in P$ so $b' = - p'' + b'xb'$. Hence

$$\begin{aligned}
n &= p' + b' = p' + (- p'' + bxb) \\
&= (p' - p'') + b'xb' \in P + BNB.
\end{aligned}$$

Thus $P + B \in P + BNB$, so $P + B = P + BNB$. Hence $P + B = P + BNB$.

Acknowledgements. The authors are very grateful to the anonymous referee for stimulating comments and improving presentation of the paper.

4. References

1) Kazim M.A. and Naseeruddin MD., On almostsemigroup, Portugaliae Mathematica, 1997;36:41- 47.

(2) Khan M., Faisal, and Amjid V., On some classes of Abel-Grassmann's groupoids, arXiv:1010.5965v2 [math.GR], 2010;2:1-6.

(3) Mushtab B. and Khan M., M-System in LA-semigroups, Southeast Asian Bulletin of Mathematics. 2009; 33: 321-327.

(4) Sarwar (Kamran) M. Conditions for LA-semigroup to resemble associative structures, [PhD thesis]. Buaid-i-Azam University; 1993.

(5) Shah T. and Rehman I., On LA-rings of finitely nonzero function, Int. J.Contemp. Math. Sciences,2010; 5: 209-222.

(6) Pilz G., Near-rings, Noth Holland, Amesterdam-New York.1997.

(7) Tamizh Chelvam T. and Ganesan N., On bi-ideals of near-rings, Indian J. Pure and Appl. Math, 1987;5:1002-1005.

(8) Choi S.J., Qusiideals of a P-regular Near-rings, International Journal of Algebra, 2010;4:501-506.

(9) Shah T., Rehman F. and Raees M., On Near Left Alomst Rings , International Mathematical Forum, 2011;6:1103-1111.

(10) Dheena P. and Manivasan S., Qusi-ideals of a P- regular Near-rings, International Journal of Algebra, 2011;5:1005-1010.