KKU Res. J. 2013; 18(6): 893-896
http : //resjournal.kku.ac.th

On Semiprime and Quasi Semiprime ideals in AG−Groupoids

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Abstract

In this paper, we study ideals, semiprime and quasi semiprime ideals in AG-groupoids. Some characterizations of semiprime ideals and quasi semiprime ideals are obtained. Moreover, we investigate relationships between semiprime and quasi semiprime ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a semiprime ideals to be a quasi semiprime ideals in AG-groupoids.

Keywords: AG-groupoid, AG-3-band, quasi semiprime ideal, semiprime ideal, right alternative.

AMS Subject Classification: 20M10, 20N99

1. Introduction

A groupoid S is called an Abel-Grassmann's groupoid, abbreviated as an AG-groupoid, if its elements satisfy the left invertive law (1, 2), that is: $(ab)\ c = (cb)\ a$ for all $a,b,c \in S$. Several examples and interesting properties of AG-groupoids can be found in (3-5) and (6). It has been shown in (3) that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. It is also known (2) that in an AG-groupoid S, the medial law, that is,

$$(ab)\ (cd) = (ac)\ (bd)$$

for all $a,b,c,d \in S$ holds. An AG-groupoid S is called AG-3-band (7) if its every element satisfies $a(aa) = (aa)\ a = a$.

Now we define the concepts that we will used. Let S be an AG-groupoid. By an AG-subgroupoid of S (8), we means a non-empty subset A of S such that $A^2 \subseteq A$. A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S (7) if $SA \subseteq A$ $(AS \subseteq A)$. By two-sided ideal or simply ideal, we mean a non-empty subset of an AG- groupoid S which is both a left and a right ideal of S. A proper ideal P of an AG-groupoid S is called semiprime (8) if $A^2 \subseteq P$ implies that $A \subseteq P$, for all ideals A in S. A proper left ideal P of an AG-groupoid S is called quasi semiprime (8) if $A^2 \subseteq P$ implies that $A \subseteq P$, for all left ideals A in S. It is easy to see that every quasi semiprime ideal is semiprime.

In this paper we characterize the AG-groupoid. We investigate relationships between semiprime and quasi semiprime ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a semiprime ideal to be a quasi semiprime ideal in AG-groupoids.

2. Basic results

In this section we refer to (7, 8) for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Lemma 2.1. (8) If S is an AG-groupoid with left identity, then every right ideal is an ideal.

Lemma 2.2. (8) If A is a left ideal of an AG-groupoid S with left identity, then aA is a left ideal in S, where $a \in S$.

Lemma 2.3. (8) If A is a right ideal of an AG-groupoid S with left identity, then A^2 is an ideal in S.

Lemma 2.4. (8) An ideal A of an AG-groupoid S is prime if and only if it is semiprime and strongly irreducible.

Lemma 2.5. (7) A subset of an AG-3-band is a right ideal if and only if it is left.

Lemma 2.6. (8) If S is an AG-groupoid with left identity, then a left ideal P of S is quasi semiprime if and only if $a(Sa) \subseteq P$ implies that $a \in P$, where $a \in S$.

Lemma 2.7. (8) If A is a proper left (right) ideal of an AG-groupoid S with left identity e, then $e \notin A$

3. Ideals in AG-groupoids

The results of the following lemmas seem to be at the heart of the theory of AG-groupoids; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 3.1. Let S be an AG-groupoid with left identity, and let B be a left ideal of S. Then $AB = \{ab : a \in a, b \in B\}$ is a left ideal in S, where $\varnothing \neq A \subseteq S$.

Proof. Suppose that S is an AG-groupoid with left identity. Let B be a left ideal of S. Then $S(AB) = A(SB) \subseteq AB$. By Definition of left ideal, we get AB is a left ideal in S.

Lemma 3.2. Let S be an AG-groupoid with left identity and let $a \in S$. Then a^2S is an ideal in S.

Proof. By Lemma 2.2, we have a^2S is a left ideal of S. Now consider

$$\begin{aligned}
(a^2r)s &= ((aa)r)s \\
&= ((ra)a)s \\
&= [e((ra)a)]s \\
&= [s((ra)a)]e \\
&= [(ra)(sa)]e \\
&= [((sa)a)r]e \\
&= [((aa)s)r]e \\
&= [(rs)(aa)]e \\
&= [ea^2](rs) \\
&= a^2\,(rs) \in a^2S
\end{aligned}$$

for all $r, s \in S$. Therefore a^2S is an ideal in S.

Lemma 3.3. Let S be an AG-groupoid with left identity, and let A, B be left ideals of S. Then $(A : B)$ is a left ideal in S, where $(A : B) = \{r \in S : Br \subseteq A\}$.

Proof. Suppose that S is an AG-groupoid. Let $s \in S$ and let $a \in (A : B)$. Then $Ba \subseteq A$ so that $B(sa) \subseteq s(Ba) \subseteq sA \subseteq A$. Therefore $sa \in (A : B)$ so that $S(A : B) \subseteq (A : B)$. Hence $(A : B)$ is a left ideal in S.

Lemma 3.4. Let S be an AG-groupoid with left identity, and let A be a left ideal of S. Then $(A : r)$ is a left ideal in S, where $(A : r) = \{a \in S : ra \in A\}$ and $r \in S$.

Proof. By Lemma 3.3, we have $(A : r)$ is a left ideal in S.

Corollary 3.5. Let S be an AG-3-band with left identity, and let A be a left ideal of S. Then $(A : r)$ is an ideal in S where $r \in S$.

Proof. By Lemma 3.4, we have $(A : r)$ is a left ideal in S. By Lemma 2.5, it follows that $(A : r)$ is a right ideal in S. By Lemma 2.1, we have $(A : r)$ is an ideal in S.

Remark 1. Let S be an AG-groupoid and let A be a left ideal of S. It is easy to verify that $A \subseteq (A : r)$.

2. Let S be an AG-groupoid with left identity e, and let A be a proper left (right) of S. By Lemma 2.7, we have $e \notin (A : r)$, where $r \in S–A$.

3. Let be San AG-groupoid and let A, B, C be left ideals of S. It is easy to verify that $(A : C) \subseteq (A : B)$, where $B \subseteq C$.

Corollary 3.6. Let S be an AG-3-band with left identity, and let A, B be left ideals of S. Then $(A : B)$ is an ideal in S.

Proof. This follows from Lemma 3.5.

4. Properties of quasi semiprime ideals in AG-groupoids.

We start with the following theorem that gives a relation between semiprime and quasi semiprime ideal in AG-groupoid. Our starting points is the following lemma:

Lemma 4.1. If S is an AG-groupoid with left identity, then a left ideal P of S is quasi semiprime if and only if $(Sa)^2 \subseteq P$ implies that $a \in P$, where $a \in S$.

Proof. Let P be a quasi semiprime left ideal of an AG-groupoid S with left identity. Now suppose that $(Sa)^2 \subseteq P$. Then by Definition of left ideal, we get

$$\begin{aligned} (Sa)^2 &= (Sa)(Sa) \\ &= (SS)(aa) \\ &= S(aa) \\ &= a(Sa) \end{aligned}$$

that is $a(Sa) = (Sa)(Sa) \subseteq P$. By Lemma 2.6, we have $a \in P$.

Conversely, assume that if $(Sa)^2 \subseteq P$, then $a \in P$ for all $a \in S$. Let $a(Sa) \subseteq P$. Now consider $a(Sa) = (Sa)^2 \subseteq P$. By using given assumption, if $a(Sa) \subseteq P$, then $a \in P$. Then by Lemma 2.6, we have P is a quasi semiprime ideal in S.

Theorem 4.2. If S is an AG-groupoid with left identity, then a left ideal P of S is quasi semiprime if and only if $a^2 \in P$ implies that $a \in P$, where $a \in S$.

Proof. Let P be a left ideal of an AG-groupoid S with identity. Now suppose that $a^2 \in P$. Then by Definition of left ideal, we get

$$\begin{aligned} (Sa)^2 &= (Sa)(Sa) \\ &= (SS)(aa) \\ &= S(aa) \\ &\subseteq SP \\ &\subseteq P. \end{aligned}$$

By Lemma 4.1, we have $a \in P$.

Conversely, the proof is easy.

Definition 4.3. (9) An AG-groupoid S is called *right alternative* if it satisfies the identity, $a(bb) = (ab)b$ for all $a,b \in S$.

Theorem 4.4. Let S be a right alternative, and let A be a quasi semiprime ideal of S. Then $(A : r)$ is a quasi semiprime ideal in S, where $r \in S$.

Proof. Assume that A is a quasi semiprime ideal of S. By Lemma 3.3, we have $(A : r)$ is a left ideal in S. Let $a^2 \in (A : r)$ Then $ra^2 \in A$ so that

$$\begin{aligned} (ra)^2 &= (ra)(ra) \\ &= r((ra)a) \\ &= r(r(aa)) \\ &= r(ra^2) \in rA \subseteq A. \end{aligned}$$

By Theorem 4.2, we have $ra \in A$ Therefore $a \in (A : r)$ and hence $(A : r)$ is a quasi semiprime ideal in S.

Corollary 4.5. Let S be a right alternative, and let A be a quasi semiprime ideal of S. Then $(A : r^2)$ is a quasi semiprime ideal in S, where $r \in S$.

Proof. This follows from Theorem 4.4.

Theorem 4.6. Let S be an AG-3-band with left identity. Then P is a quasi semiprime ideal in S if and only if P is a semiprime ideal in S.

Proof. The proof is easy.

Theorem 4.7. Let S be an AG-groupoid with left identity, and let P be a semiprime ideal of S. If $(Sa^2)^2 \subseteq P$, then $a^2 \in P$, where $a \in S$.

Proof. Let P be a semiprime ideal of an AG-groupoid S with identity. Now suppose that $(Sa^2)^2 \subseteq P$. Then by Definition of left ideal, we get

$$
\begin{aligned}
(Sa^2)^2 &= (Sa^2)(Sa^2)\\
&= ((Sb^2)a^2)S\\
&= ((a^2a^2)S)S\\
&= (SS)(a^2a^2)\\
&= a^2((SS)a^2)\\
&= a^2((a^2S)S)\\
&= (a^2S)(a^2S)\\
&= (a^2S)^2
\end{aligned}
$$

that is $(a^2S)^2$ By Lemma 3.2, we have a^2S is an ideal in S. Therefore

$$
\begin{aligned}
a^2 &= aa\\
&= (ea)a\\
&= (aa)e\\
&= a^2e \in a^2S \subseteq P.
\end{aligned}
$$

Corollary 4.8. Let S be an AG-groupoid with left identity, and let P be a semiprime ideal of S. If $(a^2)^2 \in P$, then $a^2 \in P$, where $a \in S$.

Proof. Let P be a semiprime ideal of an AG-groupoid S with identity. Now suppose that $(a^2)^2 \in P$. Then by Definition of left ideal, we get

$$
\begin{aligned}
(a^2S^2) &= (a^2S)(a^2S)\\
&= a^2((a^2S)S)\\
&= a^2((SS)a^2)\\
&= (SS)(a^2a^2)\\
&= S(a^2)^2 \subseteq SP \subseteq P
\end{aligned}
$$

that is $(a^2S^2) \subseteq P$. It is easy to see that $a^2 \in P$.

5. Conclusions

Many new classes of AG-groupoids have been discovered recently. All this has attracted researchers of the field to investigate these newly discovered classes in detail. This current article investigates the ideals, semiprime and quasi semiprime ideals in AG-groupoids. Some characterizations of semiprime ideal and quasi semiprime ideals are obtained. Moreover, we investigate relationships between semiprime and quasi semiprime ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a semiprime ideal to be a quasi semiprime ideals in AG-groupoids.

6. Acknowledgement

The authors are very grateful to the anonymous referee for stimulating comments and improving presentation of the paper.

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