Attractor of a Global Shallow Water Equation Model

ตัวดึงดูดของแบบจำลองสมการน้ำตื้นทั่วโลก

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Abstract

In this research, attractors in a global shallow water equation model are investigated using numerical experiments. The model uses a simplified topography of the earth. The variables investigated are velocity, temperature and pressure.

บทคัดย่อ

ในงานวิจัยนี้ จะศึกษาตัวดึงดูดของแบบจำลองสมการน้ำตื้นทั่วโลก โดยใช้การทดลองเชิงตัวเลข แบบจำลองมีการวิเคราะห์ทางลักษณะภูมิประเทศของโลกอย่างง่าย ตัวแปรที่ศึกษา คือ อัตราเร็ว อุณหภูมิ และความดัน

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คำสำคัญ: สมการน้ำตื้น ตัวดึงดูด

Introduction

Fluid flow is a process that can be found at any place and at any time. Examples are the flows of water in rivers, lakes, and oceans. Another important example is atmospheric flow which causes winds. The complete set of equations that describes fluid flow, the Navier-Stokes equations, is very complex which makes it impractical to apply in the real world. Thus, it is necessary to simplify the equations to applicable forms. The shallow water equations, a simplified version of the Navier-Stokes equations, are widely used in many fields including atmospheric and oceanographic studies. For large scale flow such as circulations of the atmosphere and oceans, the spherical shape of the earth must be taken into account when formulating the equations. This results in spherical shallow water equations, which are important for atmospheric and oceanic numerical model development and applications. For example, they are a basic model for atmospheric prediction, climate change study and pollution dispersion in the atmosphere.

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Although simplified from the full set of equations, the shallow water equations still maintain an important property, nonlinear behavior. As a consequence, the shallow water equations can sometimes behave in a chaotic manner that is unpredictable. In order to gain more understanding about chaos in the system, a main characteristic of chaos in a spherical shallow water equations, the attractor, will be investigated in this study.

Attractor (Kurta, 2003)

The atmospheric model is one type of dynamical system. The attractor is a characteristic of dynamical systems.

The term attractor is difficult to define in a rigorous way. An attractor is a set in the phase space that has a neighborhood in which every point stays nearby and approaches the attractor as time goes to infinity.

If the system is deterministic, its state x_n at time n uniquely determines its state x_{n+1} at time $n+1$, so there exists a mapping F, such that $x_{n+1} = F(x_n)$. The sequence $(x_n)_{n\geq 0}$ is called the *trajectory* of x_0. We can write $x_n = F^n(x_0)$, where F^n is the composition of F with itself n times. It is called the *n-th iteration* of F.

$$F^n = \overbrace{F \circ \ldots \circ F}^{n},\ F^0(x) = x,\ F^{n+1}(x) = F^n(F(x)).$$

We denote by $d(x, y)$ the distance of points x and y. The open ball with center x and radius $\delta > 0$ is the set of points whose distance from x is smaller than δ,

$$B_\delta(x) = \{y \in X : d(y, x) < \delta\}.$$

Definition 1. A dynamical system is a pair (X, F) where X is a compact metric space and $F : X \to X$ is a continuous mapping.

Definition 2. Let (X, F) be a dynamical system and $Y \subseteq X$. Y is an attractor if it is nonempty, closed, $F(Y) = Y$ and if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for any $x \in X$

$$d(x,Y) < \delta \Leftrightarrow \forall n \geq 0, d\left(F^n(x),Y\right) < \varepsilon$$

$$\& \lim_{n\to\infty} d\left(F^n(x),Y\right) = 0$$

Example1. Consider $d^2y/dt^2 + \upsilon dy/dx + \omega^2 y = 0.$ A typical trajectory in the phase space ($x^{(1)} = y$, $x^{(2)} = dy/dt$) is shown in Figure 1. We see that, as time goes on, the orbit spirals into the origin, and this is true for any initial condition. Thus, in this case the origin, $x^{(1)} = x^{(2)} = 0$, is said to be the 'attractor' of the dynamical system (Ott, 1993).

Example2. Figure 2 shows the case of a limit cycle (the dashed curve). The initial condition (labeled α) outside the limit cycle yields an orbit which, with time, spirals into the closed dashed curve on which it circulates in periodic motion in the $t \to \infty$ limit. Similarly, the initial condition (labeled β) inside the limit cycle yields an orbit which spirals outward, asymptotically approaching the dash curve. Thus, in this case, the dashed closed curve is the attractor (Ott, 1993).

Model

The experiments in this research are performed using a modified global shallow water equation model of Sato (Sato, 1998).

In the modified global shallow water model by Sato:

Prognostic Equations

$$\frac{\partial}{\partial t}u = -\mathrm{u}\cdot\nabla u - g\frac{\partial}{\partial x}Z + fv + \nu\nabla_H^2 u \qquad (1)$$

$$\frac{\partial}{\partial t}v = -\mathrm{u}\cdot\nabla v - g\frac{\partial}{\partial y}Z - fu + \nu\nabla_H^2 v \qquad (2)$$

$$\frac{\partial}{\partial t}T = -\mathrm{u}\cdot\nabla T + \nu\nabla_H^2 (T - T_0) - \frac{1}{\tau_T}(T - T_0) \qquad (3)$$

$$\frac{\partial}{\partial t}p = -\nabla\left(p_0 \mathrm{u}\right) - \frac{1}{\tau_p}\left(p - p_0\right) \qquad (4)$$

Diagnostic Equation

$$Z = \frac{RT_0}{g} + \left(\frac{RT_0}{g} - h\right)\times\frac{T - T_0}{T_0} + \frac{RT_0}{g}\times\frac{p - p_0}{p_0} \qquad (5)$$

Resolution

Horizontal grid interval: 10° longitude $\times$ 10° latitude, Integration Time: 50 days, Time step: 720 sec.

Initial Conditions and Domain

u_0=0, v_0=0,

$$p_0 = 101325\exp\left(-\frac{g}{RT_0}h\right)$$

The Experiments

For this research, the Fortran compiler is Intel Fortran. The model runs on a notebook with memory of 512 MB. The output parameters from the model are temperature (T), pressure (p) and wind speed (s). In Sato's model, the model was run for 50 days with an output interval of 24 hours. In this research, the model was run for 1200 days with an output interval of 3 hours. The model was run with 3 types of topography; land and sea, land only, and sea only. The outputs from the model were plotted to consider trajectory on phase space to find attractors using MATLAB. The number of output data points of each output parameter depends on the number of days and time interval:

number of output data points = number of days * time interval of each parameter

For plotting trajectory of the model, the x-axis is temperature (T) which has the unit of Kelvin [K], the y-axis is pressure (p) which has the unit of hectopascals (hPa) and the z-axis is wind speed (s) which has the unit of m/s.

Since the new output parameters are the approximate values on grid points, before plotting the trajectory the position of grid points must be selected for consideration in each case.

In this research, grid points at the boundaries of the domain (that is grid points at positions (1, 1) and (18, 36)) and grid points at the middle of the domain (that is grid point at position (9, 18)) are selected where the notation (i, j) means that the grid point has position i along the horizontal axis and position j along vertical axis.

If the values of output parameters for each axis are very large, then a log scale is used to reduce the values of output parameters to be plotted on the phase space.

1. Studying how to plot trajectory of Lorenz system using MATLAB.

The Lorenz system is generated by the differential equation written by

$dx/dt = -10x+10y$

$dy/dt = 28x-y-xz$

$dz/dt = -8/3z+xy$

and solved by using MATLAB. Plotting trajectory is performed using MATLAB as shown in Figure 3.

2. Running modified global shallow water equation model on notebook with Intel program.

The amount of output data depends on the number of days and the time interval which is considered in each case.

Several cases are considered as shown in Table 1. Several cases are considered because the attractors must be considered when time for running the model tends to infinity.

3. Plot output in phase space

Outputs from the model are *T*, *p* and speed. The output is plotted in phase space by MATLAB.

Results

The results of running the model on a notebook for each case are as follows:

1. Ex1, Figure 4.1 shows running for 50 days every 3 hours in which the topography is land and sea. The amount of output data is 400 output data intervals. The graph plots trajectory of output data in phase space. The notation “o” is the trajectory at each time. The trajectory in Figure 4.1 looks likely to converge to some value when time increases.

2. Ex2, Figure 4.2 shows running for 50 days every 3 hours in which the topography is land and sea. The amount of output data is 400 output data intervals. The graph plots trajectory of output data in phase space. The notation “o” is the trajectory at each time. The trajectory in Figure 4.2 looks likely to converge to some value when time increases. From experiment 1, if the output data interval on each axis is very different then a log scale is used to output parameters.

3. Ex3, Figure 5 shows running for 50 days every 3 hours in which the topography is land only. The amount of output data is 400 output data intervals. The graph plots trajectory of output data in phase space. The notation “o” is the trajectory at each time. The trajectory in Figure 5 looks likely to converge to some value when time increases. The log scale is used for the output data intervals.

4. Ex4, Figure 6 shows running for 50 days every 3 hours in which the topography is sea only. The amount of output data is 400 output data intervals. The graph plots trajectory of output data in phase space. The notation “o” is the trajectory at each time. The trajectory in Figure 6 looks likely to converge to some value when time increases. The log scale is taken to output data intervals.

5. Ex5, Figure 7 shows running for 400 days every 3 hours in which the topography is land and sea. The amount of output data is 3200 output data intervals. The graph plots trajectory of output data in phase space. The notation “o” is the trajectory at each time. The trajectory in Figure 7 looks likely to converge to some value when time increases. The log scale is taken to output data intervals.

6. Ex6, Figure 8 shows running for 400 days every 3 hours in which the topography is land only. The amount of output data is 3200 output data intervals. The graph plots trajectory of output data in phase space. The notation “o” is the trajectory at each time. The trajectory in Figure 8 looks likely to converge to some value when time increases. The log scale is taken to output data intervals.

7. Ex7, Figure 9 shows running for 400 days every 3 hours in which the topography is sea only. The amount of output data is 3200 output data intervals. The graph plots trajectory of output data in phase space. The notation “o” is the trajectory at each time. The trajectory in Figure 9 looks likely to converge to some value when time increases. The log scale is taken to output data intervals.

8. Ex8, Figure 10 shows running for 800 days every 3 hours in which the topography is land and sea. The amount of output data is 6400 output data intervals. The graph plots trajectory of output data in phase space. The notation "o" is the trajectory at each time. The trajectory in Figure 10 looks likely to converge to some value when time increases. The log scale is taken to output data intervals.

9. Ex9, Figure 11 shows running for 1200 days every 3 hours in which the topography is land and sea. The amount of output data is 9600 output data intervals. The graph plots trajectory of output data in phase space. The notation "o" is the trajectory at each time. The trajectory in Figure 11 looks likely to converge to some value when time increases. The log scale is taken to output data intervals.

Conclusion

In this research, numerical experiments of a modified global shallow water equation were performed to identify possible attractors of the model. From the experiments, the trajectories seem to converge to a value when time increases. This value could be an attractor. Further study is needed in order to verify this result.

References

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Kurta, P. 2003. **Topological and Symbolic Dynamics.** Societe Mathematique de France.

Table 1. Cases considered by running the model.

	Details	Topography	Number of outputs	Graph of trajectories
Ex1	Running for 50 day every 3 hours	Land and sea	400	Fig. 4.1, Fig. 4.2
Ex2	Running for 50 day every 3 hours	Land only	400	Fig. 5
Ex3	Running for 50 day every 3 hours	Sea only	400	Fig. 6
Ex4	Running for 400 day every 3 hours	Land and sea	3200	Fig. 7
Ex5	Running for 400 day every 3 hours	Land only	3200	Fig. 8
Ex6	Running for 400 day every 3 hours	Sea only	3200	Fig. 9
Ex7	Running for 800 day every 3 hours	Land and sea	6400	Fig. 10
Ex8	Running for 1200 day every 3 hours	Land and sea	9600	Fig. 11

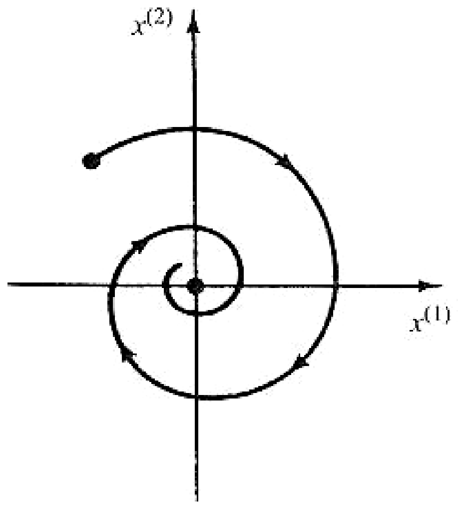

Figure 1. Attractor for Example 1.

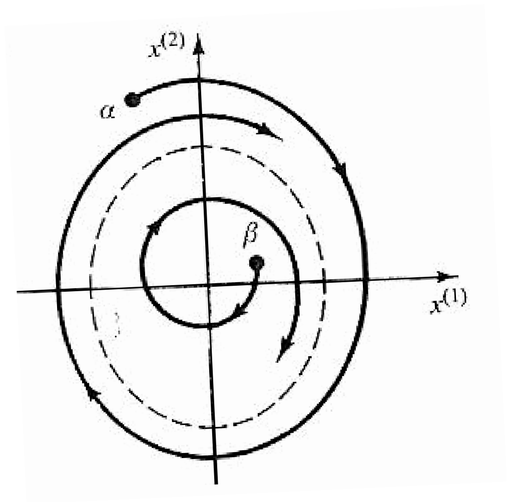

Figure 2. Attractor for Example 2.

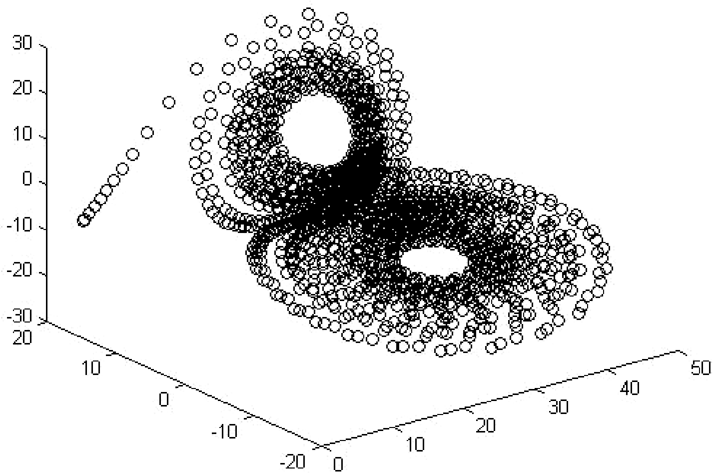

Figure 3. Lorenz Attractor.

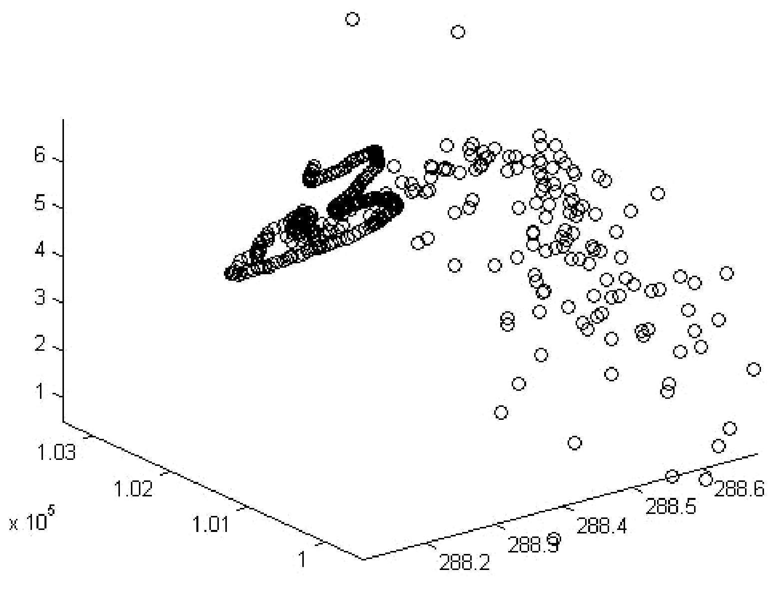

Figure 4.1 Result of Ex1. Running 50 day every 3 hour at grid point 9, 18.

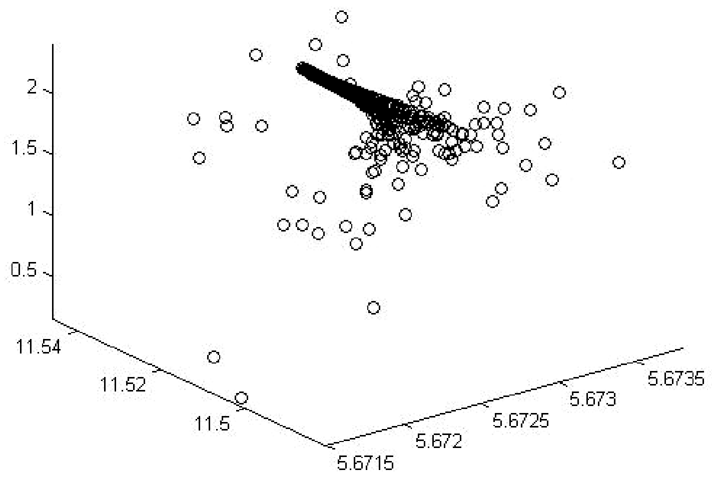

Figure 4.2 Result of Ex1. Running 50 days every 3 hours at grid point 9, 18. Take log on output.

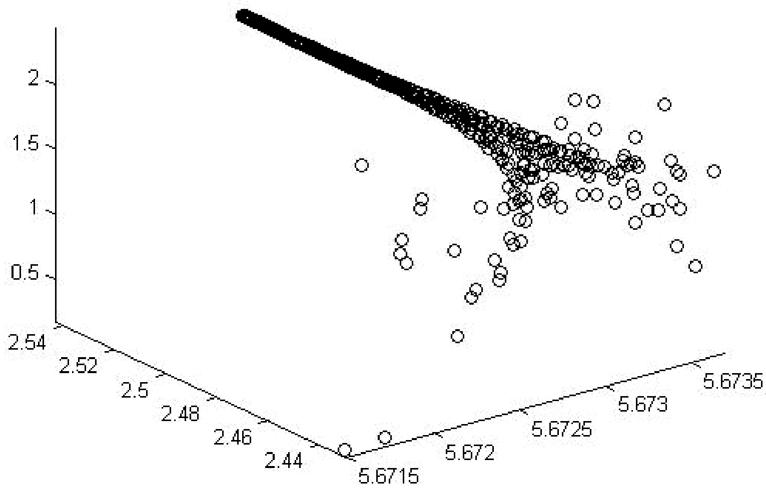

Figure 5. Result of Ex2. Running 50 days every 3 hours considering land only.

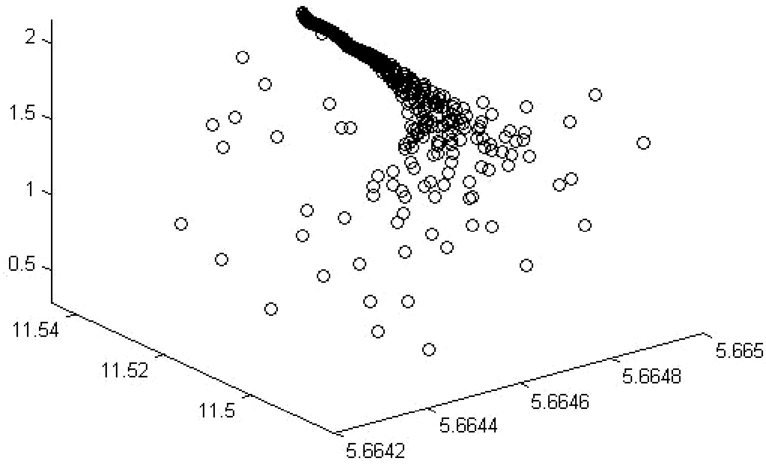

Figure 6. Result of Ex3. Running 50 days every 3 hours considering sea only.

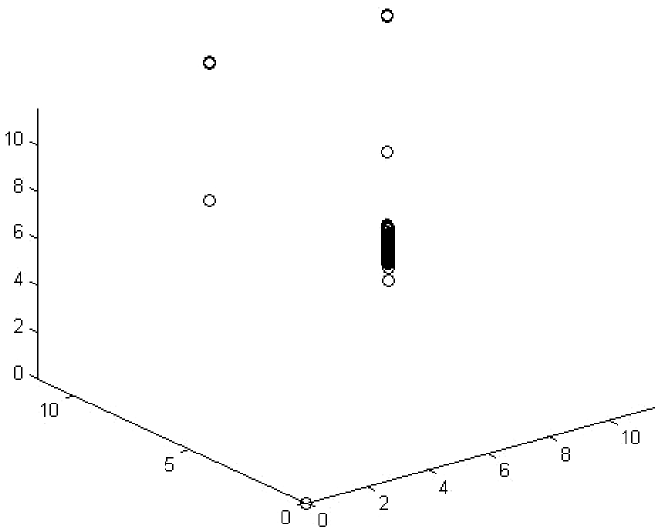

Figure 7. Result of Ex4. Running 400 days every 3 hours considering land and sea.

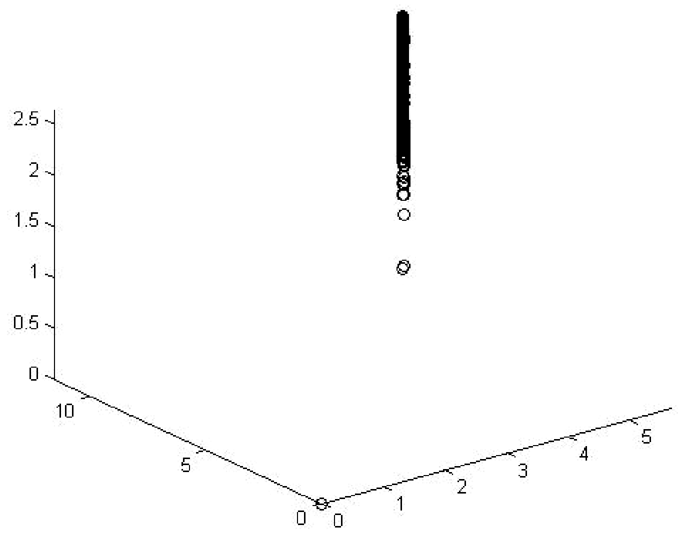

Figure 8. Result of Ex5. Running 400 days every 3 hours considering land only.

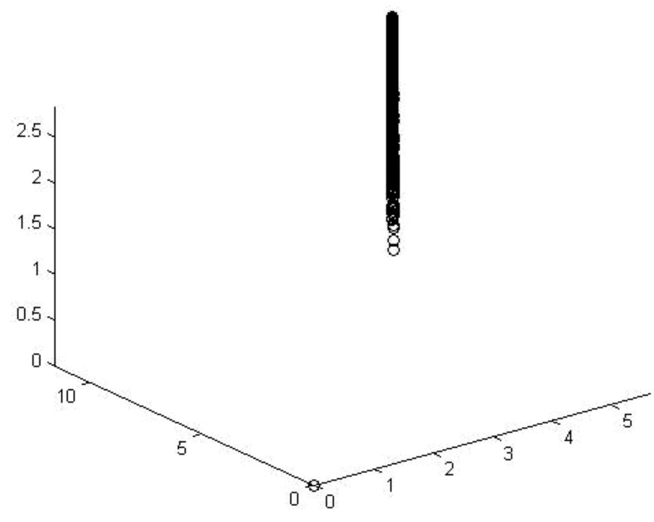

Figure 9. Result of Ex6. Running 400 days every 3 hours considering sea only.

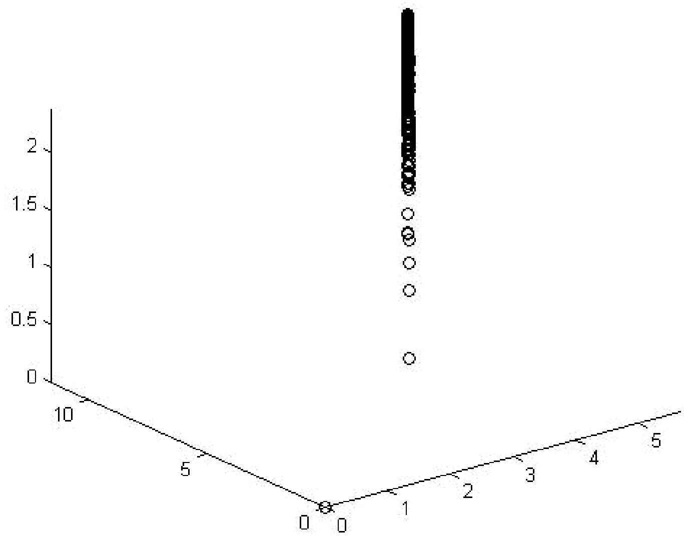

Figure 10. Result of Ex7. Running 800 days every 3 hours considering land and sea.

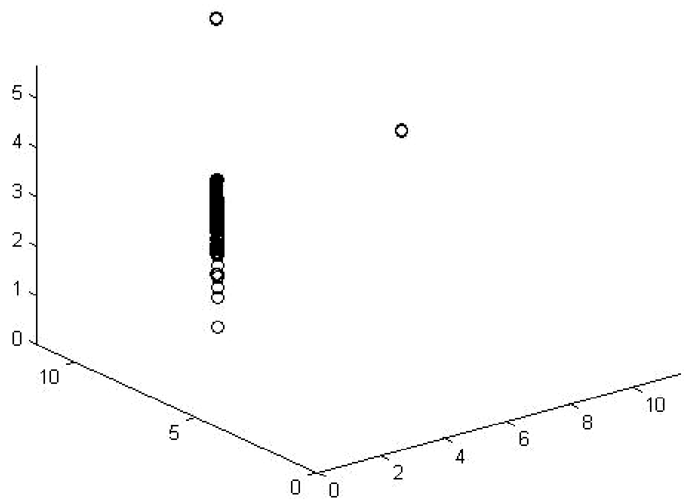

Figure 11. Result of Ex8. Running 1200 days every 3 hours considering land and sea.