Spectral Method for a Barotropic Model

ความสามารถของการทำนายในแบบจำลองบรรยากาศ

Prisayarat Sangapate (ปริศญารัตณ์ สังกะเพศ)[1]*

Dusadee Sukawat (ดุษฎี ศุขวัฒน์)[2]

Abstract

Spectral Methods are one of the most powerful solution techniques for ordinary and partial differential equations. In this research, the spectral method is applied to a barotropic model. The barotropic model is a simple model for the atmosphere. The spectral method has an advantage over finite difference methods because it has no truncation error. The spectral barotropic model is tested with a standard test case Rossby-Haurwitz wave, which has an analytic solution.

บทคัดย่อ

ศึกษาวิธีการสเปคตรัลซึ่งเป็นวิธีการหาผลเฉลยของสมการเชิงอนุพัทธ์สามัญและสมการเชิงอนุพัทธ์ย่อยที่สำคัญวิธีหนึ่งและวิธีการสเปคตรัลนี้มีข้อได้เปรียบกว่าการหาผลเฉลยของสมการเชิงอนุพัทธ์สามัญและสมการเชิงอนุพัทธ์ย่อยด้วยวิธีอื่นเพราะผลเฉลยที่ได้ไม่มีความคลาดเคลื่อนในการศึกษาครั้งนี้ได้ใช้วิธีการสเปคตรัลหาผลเฉลยของสมการอย่างง่ายทางบรรยากาศและได้เปรียบเทียบผลเฉลยของสมการอย่างง่ายทางบรรยากาศโดยใช้วิธีการสเปคตรัลกับผลเฉลยทางเชิงวิเคราะห์ งานวิจัยนี้เป็นการรายงานว่าวิธีการสเปคตรัลเป็นวิธีการหาผลเฉลยได้ใกล้เคียงกับค่าจริงมากที่สุด ซึ่งเป็นประโยชน์ต่อการนำไปประยุกต์ด้านบรรยากาศต่อไป

Keywords: Spectral method, Barotropic Model, Rossby-Hauwitz waves

คำสำคัญ: วิธีการสเปคตรัล สมการอย่างง่ายทางบรรยากาศ คลื่น

[1]*Graduate Student, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Thailand*

[2]*Lecturer, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Thailand*

corresponding author, e-mail: prisayarat1369@hotmail.com

Introduction

An atmosphere is a dynamical system, which is a system that changes over time. The barotropic model is a set of equations used to model many fluid flows. Fluid flow is a process that can be found at any place and at any time. Examples are the flows of water and air in rivers, lakes, oceans, and atmosphere. Another important example is atmospheric flow which causes winds. The barotropic model is widely used in many fields including atmospheric and oceanographic studies. This results in spherical barotropic equations, which are important for atmospheric and oceanic numerical model development and applications. For example, it is a basic model for atmospheric prediction, climate change study and pollution dispersion in the atmosphere. Thus the barotropic model is a simple model for the atmosphere. It is particularly well suited and often used to test numerical techniques for weather prediction. The spectral method is applied to a barotropic model in spherical coordinates. The spectral method has an advantage over finite difference methods because it has no truncation error. In this study, we carry through numerical standard test cases from Williamson (1992) involving the spectral barotropic model.

Model

Spectral Barotropic Model

The spectral barotropic model is a simple model of the atmosphere. The horizontal equations of the motion in Cartesian coordinates governing the non-divergent barotropic flow are

X- Momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial h}{\partial x} = 0 \qquad (1)$$

Y-Momentum:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial h}{\partial y} = 0 \qquad (2)$$

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3)$$

where u and v are horizontal velocity components, g is gravity, h is height of a pressure surface, $f = 2\Omega\sin\phi$ is the Coriolis parameter where ϕ is latitude, and Ω is the Earth angular velocity. Differentiating equation (2) by $\partial/\partial x$ and equation (1) by $\partial/\partial y$ and taking the difference between the two equations, we obtain the vorticity equation for the non-divergent barotropic flow as

$$\frac{d}{dt}\zeta_a = 0 \qquad (4)$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the relative vorticity and $\zeta_a = \zeta + f$ is the absolute vorticity. The absolute vorticity of a fluid is conserved following the motion. The velocity can be expressed in terms of a stream function ψ,

$$\mathbf{v} = \mathbf{k} \times \nabla\psi \qquad (5)$$

The relative vorticity ζ, can be expressed as follows,

$$\zeta = \nabla^2\psi \qquad (6)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian.

Thus the non-divergent barotropic vorticity equation in Cartesian coordinates can be rewritten as

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) = -J\left(\psi, \nabla^2\psi\right) - \beta\frac{\partial\psi}{\partial x} \qquad (7)$$

where $J\left(\psi, \nabla^2\psi\right)$ is the Jacobian and $\beta = \partial f/\partial y$ is the beta parameter. By using spherical coordinates, the barotropic vorticity equation in spherical coordinates can be rewritten as

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)-\frac{1}{a^2}\left[\frac{\partial\psi}{\partial\mu}\frac{\partial\nabla^2\psi}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}\frac{\partial\nabla^2\psi}{\partial\mu}\right]+\frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}=0 \quad (8)$$

where λ is longtitude and $\mu = \sin\varphi$; φ is the latitude and ∇^2 is the Laplacian such that

$$\nabla^2=\frac{1}{1-\mu^2}\frac{\partial^2}{\partial\lambda^2}+\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial}{\partial\mu}\right] \quad (9)$$

The spectral method for the barotropic vorticity equation in spherical coordinates proceeds as follows. Consider the spherical harmonic orthogonal basis functions,

$$Y_{m,n}(\lambda,\mu)=P_{m,n}(\mu)e^{im\lambda} \quad (10)$$

where $P_{m,n}(\mu)$ are associated Legendre functions, m is the zonal wave number and n is the total wave number. These basis functions are orthogonal,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} Y_{m,n}Y^*_{m',n'}\,d\mu d\lambda=\begin{cases}1, & (m,n)=(m',n')\\ 0, & (m,n)\neq(m',n')\end{cases} \quad (11)$$

From the barotropic model in spherical coordinates, the stream function can be expanded as follows

$$\psi(\lambda,\mu,t)=a^2\sum_{m=-M}^{M}\sum_{n=|m|}^{|m|+J}\psi_{m,n}(t)Y_{m,n}(\lambda,\mu) \quad (12)$$

where $\psi_{m,n}(t)$ are coefficients, M is the maximum zonal wave number, and J is the truncation meridional wave number. The vorticity is given by

$$\nabla^2\psi=-\sum_{m=-M}^{M}\sum_{n=|m|}^{|m|+J}n(n+1)\psi_{m,n}(t)Y_{m,n}(\lambda,\mu) \quad (13)$$

The spectral method is applied by substituting (12) and (13) into (8), multiplying by $Y_{m,n}^*$ and integrating with respect to μ and λ, using the orthogonality condition (11). This equation reduces to

$$\frac{\partial\psi_{m,n}(t)}{\partial t}=\frac{2\Omega mi\psi_{m,n}(t)}{n(n+1)}-\frac{1}{n(n+1)}F_{m,n} \quad (14)$$

where $F_{m,n}$ is the nonlinear advection term. Direct calculation of $F_{m,n}$ is slow because of the number of operations. It is much more efficient to Fourier transform to grid point space, evaluate the advection term there, and then transform back to wave number space (Eliasen and Orzag, 1970; Orzag, 1974). From (14), the nonlinear advection terms are

$$F(\lambda,\mu)=-\frac{1}{a}\left[\frac{1}{1-\mu^2}\frac{\partial}{\partial\lambda}\left(U\nabla^2\psi\right)+\frac{\partial}{\partial\mu}\left(V\nabla^2\psi\right)\right] \quad (15)$$

The transform of the nonlinear term $F(\lambda,\mu)$ is given by

$$F_{m,n}=-\frac{1}{2}\left[\int_{-1}^{1}\left(\frac{im}{1-\mu^2}A_mP_{m,n}+\frac{\partial}{\partial\mu}B_mP_{m,n}\right)d\mu\right] \quad (16)$$

where $P_{m,n}$ is a polynomial of degree n while A_m and B_m are coefficients. The integral can be evaluated by the Gaussian quadrature formula. Let the integrand be denoted by $Q(\mu)$, then the Gaussian quadrature formula gives the expression for $F_{m,n}$ as

$$F_{m,n}=-\frac{1}{2}\sum_{k=1}^{K}G_k^K Q(\mu_k) \quad (17)$$

where the summation is carried over K values of μ_k such that μ_k are roots of the Legendre polynomial and G_k^K are Gauss coefficients. Now $F_{m,n}$ can be computed from (17) and from (14), getting the coefficient $\psi_{m,n}(t)$ and from (12), getting the solution $\psi(\lambda,\mu,t)$ of the barotropic model.

Standard Test Case

The following test cases are proposed to evaluate and compare numerical schemes intended for global atmospheric models. Rossby-Haurwitz waves are analytic solutions of the nonlinear barotropic vorticity equation on the sphere [10]. The initial velocity field is nondivergent and given by the stream function,

$$\psi=-a^2\,\omega\sin\theta+a^2\,\mathrm{K}\cos^R\theta\sin\theta\cos R\lambda \quad (18)$$

where ω, K and R are constants. This pattern moves from west to east without change of shape in a non divergent barotropic model with angular velocity ν given by

$$\nu = \frac{R(3+R)\omega - 2\Omega}{(1+R)(2+R)} \qquad (19)$$

The velocity components are given by

$$u = a\omega\cos\theta + aK\cos^{R-1}\theta\left(R\sin^2\theta - \cos^2\theta\right)\cos R\lambda \qquad (20)$$

$$v = -aKR\cos^{R-1}\theta\sin\theta\sin R\lambda \qquad (21)$$

and the vorticity by

$$\zeta = 2\omega\sin\theta - K\sin\theta\cos^R\theta\left(R^2 + 3R + 2\right)\cos R\lambda \qquad (22)$$

The Experiments

1. *The Domain*

We take the globe and map it to a two dimensional grid in (φ,θ) coordinates where φ is the longitude and θ is the latitude.

The domain for the barotropic model in Figure 1 for this research is between 0 °W - 360 °E and -90 °N - 90 °S. It has 50 x 40 grid points in the longtitude and the latitude directions.

2. *The Variables and Time Step*

The primary variables in the barotropic model are stream function ψ and vorticity ζ. These variables are computed at each time step, with $\Delta t = 60$ second.

3. *Rossby-Haurwitz Waves*

The Rossby-Haurwitz waves test case of the barotropic model are analytic solutions of the barotropic vorticity model on the sphere. The initial velocity field is nondivergent and given by the stream function,

$$\psi = -a^2\,\omega\sin\theta + a^2 K\cos^R\theta\sin\theta\cos R\lambda$$

and the vorticity by

$$\zeta = 2\omega\sin\theta - K\sin\theta\cos^R\theta\,(R^2 + 3R + 2)\cos R\lambda$$

where $\omega = K = 7.848 \times 10^{-6}\ s^{-1}$ and R = 4 are constants. This pattern moves from west to east without change of shape in a barotropic model. Figures 2-3 illustrate initial Rossby-Haurwitz waves for stream function ψ and vorticity ζ.

Figures 4-7 show the spectral barotropic model after runs of 50 and 100 days by using the standard test case. The results look the same as the initial conditions and are preserved. Note that we have visibly achieved our goal of retaining the initial shape the Rossby-Haurwitz waves.

Conclusion

The experiments in this research were performed by using a spectral barotropic model. The spectral barotropic model is tested with a standard test case Rossby-Haurwitz wave, which has an analytic solution. The results after running the spectral barotropic model by using a standard test case Rossby-Haurwitz wave look the same as the initial condition. The Rossby-Haurwitz wave retains the initial shape. The spectral method has an advantage over finite difference methods because it has no truncation error.

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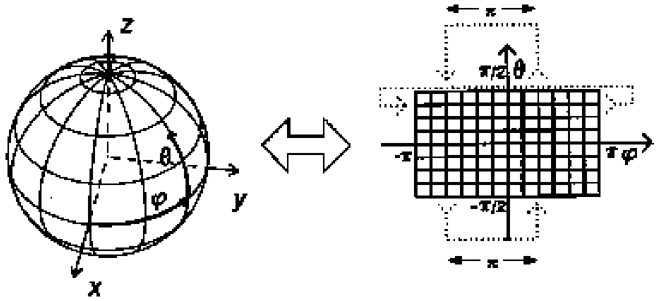

Figure 1. The domain for the barotropic model.

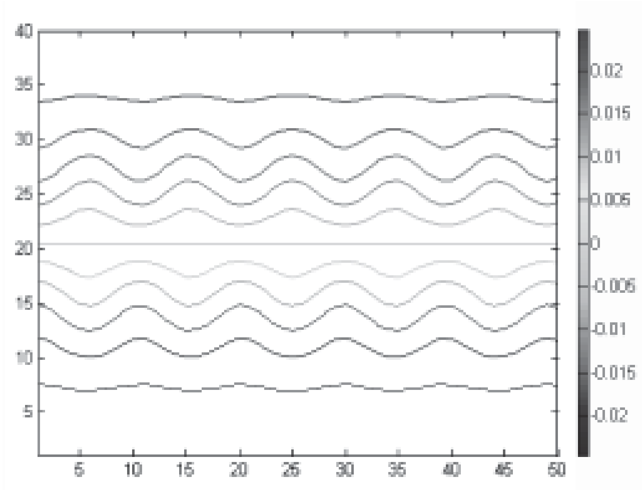

Figure 2. Initial Rossby Haurwitz waves for ψ.

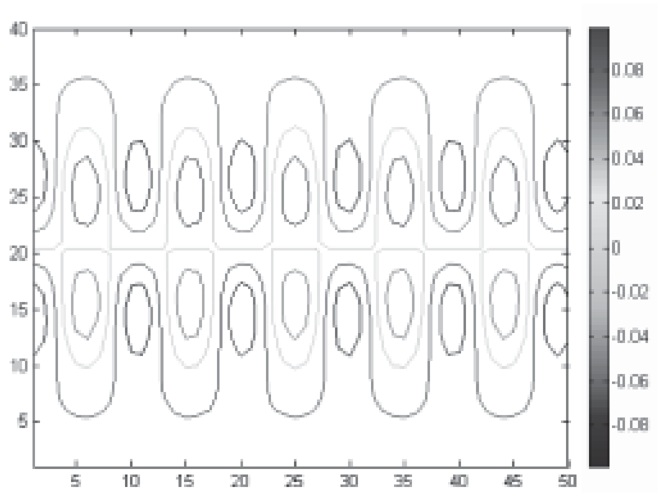

Figure 3. Initial Rossby Haurwitz waves for ζ.

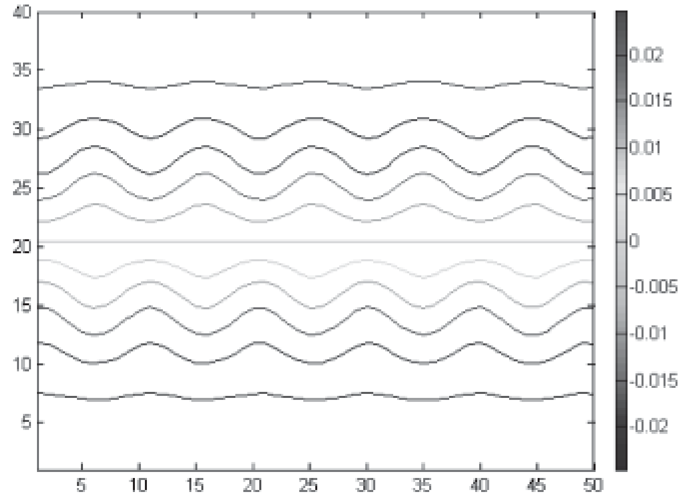

Figure 4. Rossby-Haurwitz waves for ψ after running 50 days.

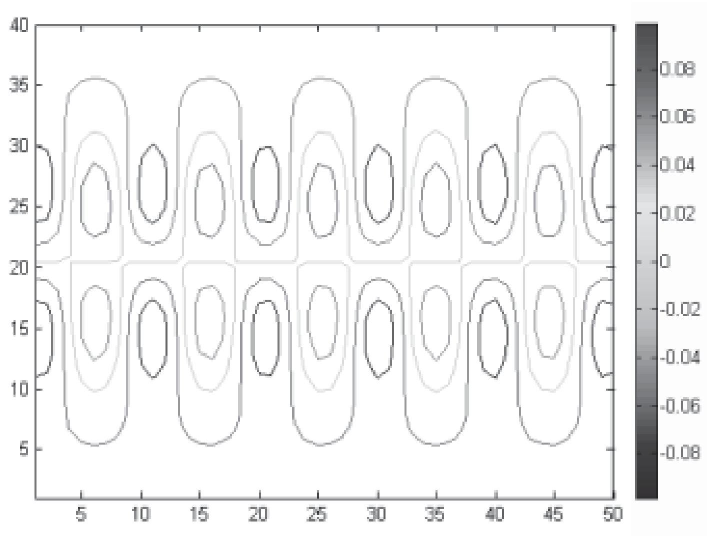

Figure 5. Rossby-Haurwitz waves for ζ after running 50 days.

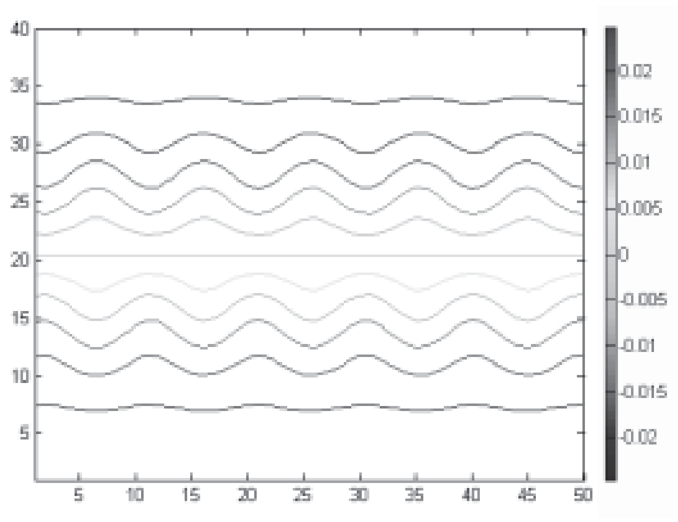

Figure 6. Rossby–Haurwitz waves for ψ after running 100 days.

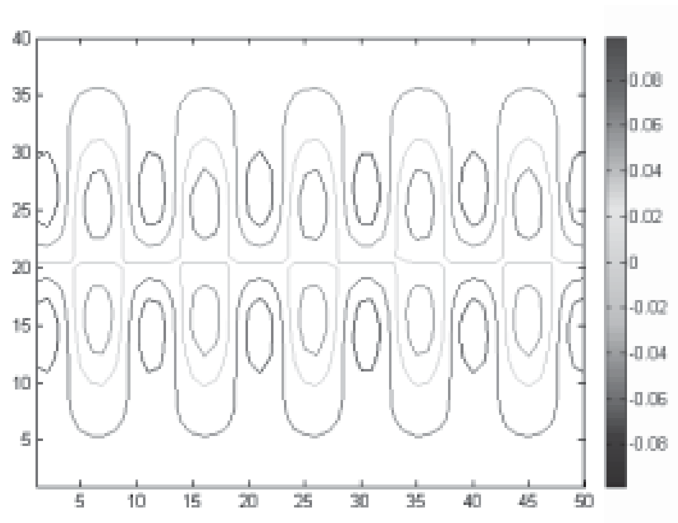

Figure 7. Rossby–Haurwitz waves for ζ after running 100 days.