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Approximate average run length and its eigenvalue problem on exponentially weighted moving average control chart: an economic application

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Abstract

The average run length (ARL) is a criterion for measuring the efficiency of a control chart conventionally computed based on the assumption of type I errors for the in-control process and type II errors for the out-of-control process. Still, the eigenvalue approach for the ARL by controlling the direction on its eigenvector is a good alternative. Thus, the objectives of this research are to evaluate the ARL based on the eigenvalue approach on an exponentially weighted moving average (EWMA) control chart and to apply ARL computation to the inflation rate data of the Thai economy. The methods used for ARL evaluation in a comparative study are based on integral equations, a numerical method, the eigenvalue approach, parameter estimation, and fitting of the probability density function. The findings show that the distinct eigenvalues of the ARL on an EWMA control chart monitoring the Thai economy inflation rate with a symmetric kernel are all real and the maximum eigenvalue returns the maximum values of ARL0 (the in-control process) and ARL1 (the out-of-control process). Moreover, an eigenvalue close to zero returns ARL0 and ARL1 values close to one.

Keywords: Average run length, Exponentially weighted moving average control chart, Integral equation, Eigenvalue problem, Inflation rate data of the Thai economy

1. Introduction

Statistical process control based on decision-making, statistical theory, and sequential processing is used to monitor unexpected occurrences in industrial processes. Moreover, estimating and controlling the inflation rate of a national economy by using statistical process control is an interesting topic for investment and administration. A tool widely used in statistical process control is the exponentially weighted moving average (EWMA) control chart. Measuring control chart efficiency is achieved using the average run length (ARL), which is the mean of the number of observations before the first out-of-control observation. Methods for evaluating the ARL are mainly computed in three approaches: Markov chain, Monte Carlo simulation, and integral equations [1,2,10,16,17].

Control charts have been developed by many researchers. Shewhart's control chart, initiated and released by [11], can detect large changes in the process mean when the observations follow a normal distribution. Moreover, it consists of control limits with the three-sigma value of the standard deviation of the process. However, in practice, observations do not necessarily follow a normal distribution because of criteria such as trends, seasonal variation, and autocorrelation. Hence, EWMA was developed by [12] as a statistic for control charts when the data have a trend and/or seasonality, especially when the observations change gradually. Furthermore, the EWMA control chart is advantageous for detecting small shifts in the process mean.

The ARL can be derived in the form of an integral equation of the second kind [4,6,16]. There have been several studies on integral equations for evaluating the ARL and comparing the efficiencies of control charts based on the ARL. An EWMA control chart for detecting a change in the observations via the Martingale approach for normal and non-normal distributions was proposed by [14]. In [1], the Martingale approach was used for the ARL of an EWMA control chart process for changes in an exponential distribution, and its effectiveness was compared on EWMA and cumulative sum (CUSUM) control charts with an autoregressive process; evaluating the ARL via the Martingale approach is a modern approach that required less computational time than a simulation approach. The ARL integral equation method for an EWMA control chart with given observations following a log-normal distribution was estimated by [12] and compared with the ARL using a simulation method. Integral equations were applied by [10] to determine the ARLs on an EWMA control chart with observations following a Laplace distribution and on a CUSUM control chart with observations following a hyperexponential distribution; the analytical ARLs were precise and required less computational time than the simulation method. [17] investigated analytical and numerical ARLs for an EWMA control chart with a long memory autoregressive fractionally integrated moving average (ARFIMA) process and compared their performances; the results showed that the analytical ARL could be computed quicker than the numerical ARL, thereby showing the former to be a good alternative for evaluating the efficiency of a control chart. An adaptive EWMA control chart with a time-varying smoothing parameter was proposed by [19]; the ARL was calculated on a discrete-time Markov chain with an optimal smoothing parameter and controlling the width of the control chart to minimize the ARL by using a nonlinear optimization method. Moreover, a comparison of the ARLs of the adaptive EWMA control chart and traditional control charts showed the benefits of the proposed research.

Inflation in the Thai economy determined by changes in the prices of products and services is classified into the core inflation rate and the headline inflation rate. A high inflation rate is the result of high production costs and retail prices. We examined the core inflation rate data reported by the Bureau of Trade and Economic Indices in the Ministry of Commerce of Thailand and the Bank of Thailand [3]. Thus, some researchers have studied the inflation rate of the Thai economy. Forecasting the core inflation rate using a method derived from short-term and long-term properties of time series was reported by [15]; the forecasting error is based on the root-mean-squared and mean-absolute errors. The effect of the inflation rate on stock prices in Thailand from January 2000 to March 2010, a period involving the tsunami in 2004 and the global financial crisis in 2008, was researched by [9]; their findings are that the inflation rate and stock price movements were not related during this time period. The relationship between monetary policy rules and the inflation rate in Thailand since 2000 was studied by [18]; their findings show that the inflation rate gradually responded to the monetary policy rule over the long-term.

Research related to estimating parameters for fitting distributions to inflation rate data are discussed here. Evaluation and risk analysis of a gold mine project in Iran were studied by [13]. The feasibility of the gold mine project was based on the net present value or economic valuation. They concluded that the net present value of the project was the most sensitive variable to gold and silver prices, while other variables such as discount rate and operating and capital costs all affected the feasibility of the project to a lesser extent. Meanwhile, their results also show that the inflation rate in Iran followed a logistic distribution with estimates of 0.154157 for the location parameter and 0.029245 for the scale parameter. Fitting of the monthly inflation rate distribution of the Nigerian economy from 1997-2014 was demonstrated by [5]. Inflation rate data collected from the Central Bank of Nigeria were fitted appropriately to a three-parameter log-logistic distribution and a logistic distribution, as was confirmed via a Kolmogorov Simonov goodness-of-fit test; the parameters of both distributions were estimated by applying a maximum likelihood method.

An efficient control chart must quickly detect observations indicating an out-of-control process. Eigenvalues can solve this problem because they can determine the direction of the ARL, which is the measurement criterion for the efficiency of a control chart covering three standard deviations of observations. Determining the ARL direction can lead to obtaining an efficient control chart. Therefore, an integral equation method for evaluating the ARL and its eigenvalue on an EWMA control chart was applied to inflation rate data from the Thai economy.

2. Materials and methods

This section presents the definitions of mathematical and statistical backgrounds research as well as the methodology of this research.

2.1 Preliminaries and definitions

Definition 1 Exponentially Weighted Moving Average (EWMA) control chart is based on EWMA statistic Y_t starting with initial value $Y_0 = \mu = w$ and λ is smoothing parameter with observation X_t with mean μ and variance σ^2 as

$$Y_t = (1-\lambda)Y_{t-1} + \lambda X_t \quad ; t = 1,2,3,...,n; \, Y_0 = \mu \tag{1}$$

Definition 2 average run length (ARL) is the expected value of the number of observations until the first detected observation out of control. The mathematical definition for ARL is defined a stopping time with initial value $\mu = w$ of statistic Y_t with the lower bound M and upper bound N of EWMA control chart as

$$ARL(w) = \mathrm{E}_\infty(\tau_{M,N}) \text{ where } \tau_{M,N} = \inf\{t > 0 : Y_t < M \text{ or } Y_t > N\}$$

Also, ARL can be classified to two stages: the ARL for in control process state denoted as ARL0 and the ARL for out of control process state denoted as ARL1.
Definition 3 the second kind of integral equation for ARL [8] is given as

$$ARL(w) = g(w) + \kappa \int_M^N K(w,y) ARL(y) dy \tag{2}$$

where $K(w,y)$ is a kernel function, $g(w)$ is a given function, and $ARL(w)$ is an unknown ARL function, κ is an eigenvalue.

Definition 4 let $K(w,y)$ be an infinitely differentiable function of two variables about $(w,y) = (M,N)$, the Taylor' series for a function of two variables can be defined as

$$\begin{aligned} K(w,y) &= K(M,N) + K_w(M,N)(w-M) + K_y(M,N)(y-N) \\ &+ \frac{1}{2!}[K_{ww}(M,N)(w-M)^2 + 2K_{wy}(M,N)(w-M)(y-N) + K_{yy}(M,N)(y-N)^2] + ... \end{aligned} \tag{3}$$

Definition 5 let A be an $n \times n$ matrix. The scalar κ is called eigenvalue of A if there exist vector $\vec{x} \neq \vec{0}$ such that

$$A\vec{x} = \kappa \vec{x}. \tag{4}$$

where $\vec{x}$ is called eigenvector corresponding to eigenvalue κ of matrix A.

Definition 6 a complex inner product space is a complex vector space V with inner product (denoted by $< x, y >$) which a function or mapping $: V \times V \to C$ satisfying the followings

1) $< x,x >> 0$; $< x,x >= 0$ if $x = 0$ for $\forall x \in V$,
2) $< \alpha x + \beta x, z >= \alpha < x,z > + \beta < y,z >$ for $\forall \alpha, \beta \in C$,
3) $\forall x,y,z \in V$, $< y,x >= \overline{< x,y >}$ for $\forall x,y \in V$.

Also, the norm of inner product space is defined as norm = $\|x\| = \sqrt{< x,x >}$; $x \in V$. A complex inner product vector space with respect norm on complex vector space V is called a Hilbert space.

2.2 Methodology

2.2.1 Proposed integral equation for the average run length for EWMA control chart

EWMA statistic is firstly presented by [12] as $Y_t = (1-\lambda)Y_{t-1} + \lambda X_t \quad ; t = 1,2,3,...,n$

with $Y_1=(1-\lambda)Y_0+\lambda X_1$; $Y_0=\mu$. For in control state at time t, $M\le X_t\le N$ with an initial control state $Y_0=\mu=w$ such that $M\le w\le N$; but out of control state at time t, $Y_t<M$ or $Y_t>N$. For ARL for initial value w, $ARL(w)=E_\infty(\tau_{M,N})$.

For this research, the proposed integral equation for evaluating ARL is developed from the researches of [4] and [6]. The EWMA statistic is adapted to scale of observation X_t with a scale factor $\kappa\ne 0$. That is, let $\frac{1}{\kappa}X_1,\frac{1}{\kappa}X_2,\ldots,\frac{1}{\kappa}X_n$ be independent and identically distributed random variables with a positive real $\kappa>0$ as follows:

$$Y_t=(1-\lambda)Y_{t-1}+\lambda\frac{1}{\kappa}X_t \quad ; t=1,2,3,\ldots,n$$

Thus, there are two possibilities for Y_1 after the first observation X_1 is made.

Case 1: If X_1 gives an out of control for Y_1, then $Y_1<M$ or $Y_1>N$. Namely $(1-\lambda)Y_0+\lambda\frac{1}{\kappa}X_1<M$ or $(1-\lambda)Y_0+\lambda\frac{1}{\kappa}X_1>N$. In this case, the run length will be 1 because there will be an immediate out of control signal.

Case 2: If X_1 gives Y_1 in an in control state, then $M\le Y_1\le N$ Namely, $(\kappa)\frac{M-(1-\lambda)w}{\lambda}\le X_1\le(\kappa)\frac{N-(1-\lambda)w}{\lambda}$

In this case, an observation will have been made an on average $ARL(Y_1)=ARL((1-\lambda)w+\frac{\lambda}{\kappa}X_1)$ Also, more observations will be made before an out of control signal occurs. The probability density function for X_1 is given as $f(X_1)$.

Let $\partial\Omega$ be a domain of in control state: $\{M\le(1-\lambda)w+\frac{\lambda}{\kappa}X_1\le N\}$. Therefore, the expected run length can be formulate as

$$ARL(w)=1\cdot\Pr(|(1-\lambda)w+\frac{\lambda}{\kappa}X_1|>N)+\int_{\partial\Omega}[1+ARL((1-\lambda)w+\frac{\lambda}{\kappa}x)]f(x)dx \tag{5}$$

$$ARL(w)=1+\int_{(\kappa)\frac{M-(1-\lambda)w}{\lambda}}^{(\kappa)\frac{N-(1-\lambda)w}{\lambda}}[ARL((1-\lambda)w+\frac{\lambda}{\kappa}x)]f(x)dx$$

Changing variable, $y=(1-\lambda)w+\frac{\lambda}{\kappa}x$ Therefore, the integral equation representing ARL is given as

$$ARL(w)=1+\kappa\int_M^N\frac{1}{\lambda}f(\kappa(\frac{y-(1-\lambda)w}{\lambda})ARL(y)dy;\ \kappa>0 \tag{6}$$

Similarly, in case of a negative real $\kappa<0$, $ARL(w)=1+\int_{(\kappa)\frac{N-(1-\lambda)w}{\lambda}}^{(\kappa)\frac{M-(1-\lambda)w}{\lambda}}[ARL((1-\lambda)w+\frac{\lambda}{\kappa}x)]f(x)dx$ and the integral equation representing ARL becomes $ARL(w)=1+\kappa\int_N^M\frac{1}{\lambda}f(\kappa(\frac{y-(1-\lambda)w}{\lambda})ARL(y)dy;\ \kappa<0$ (7)

In case of $\kappa=0$, obviously, $ARL(u)=1$.

2.2.2 Computation of the ARL on an EWMA control chart for application to the inflation rate of the Thai economy

In this research, the ARL can be derived in the form of integral equations of the second kind is in the form in Equation (2). The numerical integral equation is employed to solve the un-separable kernel function of the integral equation. A Taylor's series approximation is the numerical method adopted to separate the kernel and estimate the numerical ARL.

An assumption in this research is that the inflation rate data of the Thai economy follows a logistic distribution. The ARL of an EWMA control chart can be formulated as

$$ARL(w;\mu,s) = 1+\kappa\int_M^N \frac{1}{4\lambda s} ARL(y;\mu,s)\sec h^2\left[\frac{y}{2\lambda s}+\frac{(w-\mu)\lambda - w}{2\lambda s}\right])dy \tag{8}$$

where $\sec h(\cdot)$ is a hyperbolic secant function, κ is the eigenvalue of the ARL as an integral equation based on the average and standard deviation of the real inflation rate, M is 3.3, and N is 10.

The method for approximating the kernel function of the ARL integral equation is based on a Taylor's series. For this research, $K(w,y)$ is approximated as $\overline{K(w,y)}$ in Equation (3) and separable in the summation of the function of variable w and y as $\overline{K(w,y)} = \sum_{i=1}^{n} a_i(w)b_i(y)$. $\overline{ARL(w)} = g(w)+\kappa\sum_{i=1}^{n} a_i(w)\int_M^N b_i(y)ARL(y)dy.$

By multiplying both sides by $b_m(w)$; $m=1,2,...,n$ and then integrating both sides, we obtain

$$\int_M^N b_m(w)\overline{ARL(w)}dw = \int_M^N b_m(w)g(w)dw + \kappa\sum_{i=1}^{n}\int_M^N b_m(w)a_i(w)dw\int_M^N b_i(y)\overline{ARL(y)}dy.$$

$$l_m = g_m + \kappa\sum_{i=1}^{n} a_{mi}l_i;\ m=1,2,...,n \tag{9}$$

where $L_m = \int_M^N b_m(w)\overline{ARL(w)}dw$, $g_m = \int_M^N b_m(w)g(w)dw$, $a_{mi} = \int_M^N b_m(w)a_i(w)dw$, $c_i = \int_M^N b_i(y)\overline{ARL(y)}dy$.

Equation (9) is a nonhomogeneous system of n linear equations and can be written in matrix form as

$$L = G + \kappa AL \tag{10}$$

Equation (10) has a unique solution if $\det((I-\kappa A)) \neq 0$. Otherwise, if $\det((I-\kappa A)) = 0$, then there is either no solution or an infinite number of solutions of l_i, which is the eigenvector corresponding to $ARL(w;\mu,s) = g(w)+\kappa\sum_{i=1}^{n} l_i a_i(w)$. Moreover, the eigenvalue and eigenvector of the ARL integral equation are computed as a nonhomogeneous system of linear equations.

For application to the inflation rate of the Thai economy, Equation (10) can be solved by approximating the kernel method based on a Taylor's series approximation and the numerical method for evaluating the ARL as the solution of the integral equation. The kernel function $K(w,y)$ is not separable, so it can be estimated via the Taylor's series as

$$\begin{aligned}
&\overline{K(w,y;\mu,s)} = \sec h^2\left[\frac{y}{2\lambda s}+\frac{(w-\mu)\lambda-w}{2\lambda s}\right] \\
&\approx 1-(\frac{y}{2\lambda s})^2+\frac{2}{3}(\frac{y}{2\lambda s})^4-2(\frac{y}{2\lambda s})\frac{(w-\mu)\lambda-w}{2\lambda s}+\frac{8}{3}(\frac{y}{2\lambda s})^3\frac{(w-\mu)\lambda-w}{2\lambda s} \\
&-(\frac{(w-\mu)\lambda-w}{2\lambda s})^2+4(\frac{y}{2\lambda s})^2(\frac{(w-\mu)\lambda-w}{2\lambda s})^2+\frac{8}{3}(\frac{y}{2\lambda s})(\frac{(w-\mu)\lambda-w}{2\lambda s})^3 \\
&+\frac{2}{3}(\frac{(w-\mu)\lambda-w}{2\lambda s})^4
\end{aligned} \tag{11}$$

The method for solving ARL of Equation (8) by approximating the kernel is applied: $\overline{K(w,y)} = \sum_{i=1}^{7} a_i(w)b_i(y)$

where $a_1(w) = 1-(\frac{(w-\mu)\lambda-w}{2\lambda s})^2+\frac{2(\frac{(w-\mu)\lambda-w}{2\lambda s})^4}{3}$, $a_2(w) = -2(\frac{(w-\mu)\lambda-w}{2\lambda s})$, $a_3(w) = \frac{8}{3}(\frac{(w-\mu)\lambda-w}{2\lambda s})^3$, $a_4(w) = -1$, $a_5(w) = 4(\frac{(w-\mu)\lambda-w}{2\lambda s})^2$, $a_6(w) = \frac{8}{3}(\frac{(w-\mu)\lambda-w}{2\lambda s})$, $a_7(w) = \frac{2}{3}$, $b_1(y) = 1$, $b_2(y) = \frac{y}{2\lambda s}$, $b_3(y) = \frac{y}{2\lambda s}$, $b_4(y) = (\frac{y}{2\lambda s})^2$, $b_5(y) = (\frac{y}{2\lambda s})^2$, $b_6(y) = (\frac{y}{2\lambda s})^3$, $b_7(y) = (\frac{y}{2\lambda s})^4$.

Equation (8) becomes $\overline{ARL(w)} = 1+\kappa\int_M^N \sum_{i=1}^{7} a_i(w)b_i(y)ARL(y)dy$

$$\overline{ARL(w)} = 1+\kappa\sum_{i=1}^{7} a_i(w)\int_M^N b_i(y)ARL(y)dy \tag{12}$$

Therefore, $l_m = g_m + \kappa\sum_{i=1}^{7} a_{mi}\, l_i;\ m = 1,2,...,7$.

It can be written in matrix form as

$$\begin{aligned} L &= G+\kappa AL \\ L &= (I-\kappa A)^{-1}G. \end{aligned} \tag{13}$$

Therefore, for shift size δ, the ARLs for the in-control and out-of-control states are $\mathrm{ARL0}(w;\mu=\mu_0,s)$ and $\mathrm{ARL1}(w;\mu=\mu_0+\delta,s)$, respectively.

2.2.3 Convergence and error analysis

Let $(Con([M,N]),\|\cdot\|_\infty)$ be space of all continuous functions $ARL(w)$ where $w\in[M,N]$ and norm of function $ARL(w)$ is defined as $\|ARL(w)\|_\infty = \max_{x\in[M,N]}|ARL(w)|$. Let B be an upper bound for $|k_w|,|k_y|,|k_{ww}|,|k_{wy}|,|k_{yy}|$ on region: $\sqrt{(|w-M|+|y-N|)^2} < R$. Assume that $f(w)\neq 0$, $|k(w,y)|\le B;\ \forall w,y\in[M,N]$ and $\vartheta = \kappa(|B|+\frac{1}{2}BR^2)(M-N)$.

$$\begin{aligned}
&\overline{K(w,y)} = K(M,N)+K_w(M,N)(w-M)+K_y(M,N)(y-N) \\
&+\frac{1}{2!}[K_{ww}(M,N)(w-M)^2+2K_{wy}(M,N)(w-M)(y-N)+K_{yy}(M,N)(y-N)^2]+....
\end{aligned}$$

Theorem 1 Let $ARL(w)$ and $\overline{ARL(w)}$ be an analytical and an estimated solutions of Equation (8) or numerical ARL, respectively, the solutions of Equation (8) are convergent if $0<\vartheta<1$.

Proof

$$\left\|ARL(w)-\overline{ARL(w)}\right\|_{\infty} = \max_{x\in[M,N]} |[g(w)+\kappa\int_{M}^{N} K(w,y)ARL(y)dy]-[g(w)+\kappa\int_{M}^{N}\overline{K(w,y)}\,\overline{ARL(y)}dy]$$

$$\leq \kappa\int_{M}^{N} | K(w,y)-\overline{K(w,y)} || ARL(y)-\overline{ARL(y)}| \, dy$$

$$\leq \kappa\int_{M}^{N}[| K(w,y)|+\frac{1}{2}\{| w-M|^{2}| K_{ww}(M,N)|+2| w-M|| y-N|| K_{wy}(M,N)|$$

$$+| y-N|^{2}| K_{yy}(M,N)|\}]\left\|ARL(y)-\overline{ARL(y)}\right\|_{\infty} dy$$

$$\leq \kappa(| B|+\frac{1}{2}BR^{2})(N-M)\left\|ARL(y)-\overline{ARL(y)}\right\|_{\infty}$$

Thus, $$\left\|ARL(y)-\overline{ARL(y)}\right\|_{\infty} \leq \vartheta\left\|ARL(y)-\overline{ARL(y)}\right\|_{\infty}$$

where $\vartheta=\kappa(| B|+\frac{1}{2}BR^{2})(N-M)$. If $0<\vartheta<1$, then $\lim_{n\to\infty}\left\|ARL(y)-\overline{ARL(y)}\right\|_{\infty}=0.$

Therefore, the estimated solution of ARL of integral equation or numerical ARL is convergent to the analytical solution.

Corollary 1 Under the assumption of Theorem 1, the error of the estimated solution of ARL of integral equation can be evaluated and given as:

$$\left\|ARL(y)-\overline{ARL(y)}\right\|_{\infty} \leq \kappa(| B|+\frac{1}{2}BR^{2})(N-M)\left\|ARL(y)-\overline{ARL(y)}\right\|_{\infty}.$$

Next, considering the eigenvalues of ARL on Hilbert space is shown.

2.2.4 Eigenvalue of average run length on the Hilbert space

Let an operator T be a linear operator with kernel $K(w,y)$ is defined by

$$T(ARL(w))=g(w)+\kappa\int_{M}^{N} K(w,y)ARL(y)dy;\ \kappa\neq 0$$

Namely, $T(ARL(w))=\kappa ARL(w)$ for nonzero scalar factor $\kappa\neq 0$, which is the eigenvalue corresponding to eigenvector $ARL(w)$. The equation $ARL=g+\kappa T(ARL)$ forms a Hilbert Space. Let the set of vectors $\{ARL_n\}$ on the real be orthonormal on the real corresponding to the eigenvalue.

Theorem 2 Distinct eigenvalues corresponding to orthonormal eigenvectors of the integral equation representing ARL with a symmetric kernel are all real.

Proof

Obviously, $\overline{K(w,y)}=\sum_{i=1}^{n}a_i(w)b_i(y)$ is a symmetric function, and we assume that the eigenvalue of ARL is a complex number. Let ARL be defined on a complex number field as $ARL=\phi+\varphi j;\ j=\sqrt{-1}$. i.e., $(ARL)'=\phi-\varphi j$.

If $$T(ARL)=(\eta+\upsilon j)ARL=(\eta\phi-\upsilon\varphi)+(\eta\varphi+\phi\upsilon)j$$

,then $$T((ARL)')=(\eta+\upsilon j)(ARL)'=(\eta\phi+\upsilon\varphi)+(\phi\upsilon-\eta\varphi)j$$

To obtain two distinct eigenvalues, i.e., $\upsilon\neq 0$, the two eigenvectors corresponding to the two eigenvalues are orthonormal. Namely, $<ARL,(ARL)'>=0$ $\phi^2+\varphi^2=0$ i.e., for $\phi=0,\ \varphi=0$, and $ARL=0$, $(\eta+\upsilon j)$ with $\upsilon\neq 0$ cannot be an eigenvalue.

Therefore, the eigenvalues of the integral equation representing ARL based on an EWMA control chart of the inflation rate data of the Thai economy with a symmetric kernel are all real.

Theorem 3 The maximum eigenvalue returns the maximum ARL and an eigenvalue close to zero gives an ARL close to one.

Proof

From the integral equation representing ARL,

$$ARL(w;\mu,s)=1+\kappa\int_{M}^{N}\frac{1}{4\lambda s}ARL(y;\mu,s)\sec h^{2}\left[\frac{y}{2\lambda s}+\frac{(w-\mu)\lambda-w}{2\lambda s}\right])dy.$$

$$\text{Obviously, } \sec h^{2}\left[\frac{y}{2\lambda s}+\frac{(w-\mu)\lambda-w}{2\lambda s}\right]>0 \text{ .}$$

Therefore, when κ is the maximum value, the ARL is the maximum value, and when κ tends to zero, the ARL approaches one.

3. Results and discussion

The inflation rate of the Thai economy was used to test the proposed method. Estimating the parameters of the Thai economy yearly inflation rate data collected from 1979 to 2019 with 2015 as the base year is illustrated by applying the maximum likelihood method. The ARLs for the in-control and out-of-control states along with the eigenvalue and eigenvector for the ARL are reported.

Table 1 summarizes the results of the estimated location and scale parameters of the Thai economy inflation rate data distribution and testing whether it fits a logistic distribution using the maximum likelihood function and the Kolmogorov-Smirnov method, respectively. The probability density function of the inflation rate data of the Thai economy is demonstrated in Figure 1.

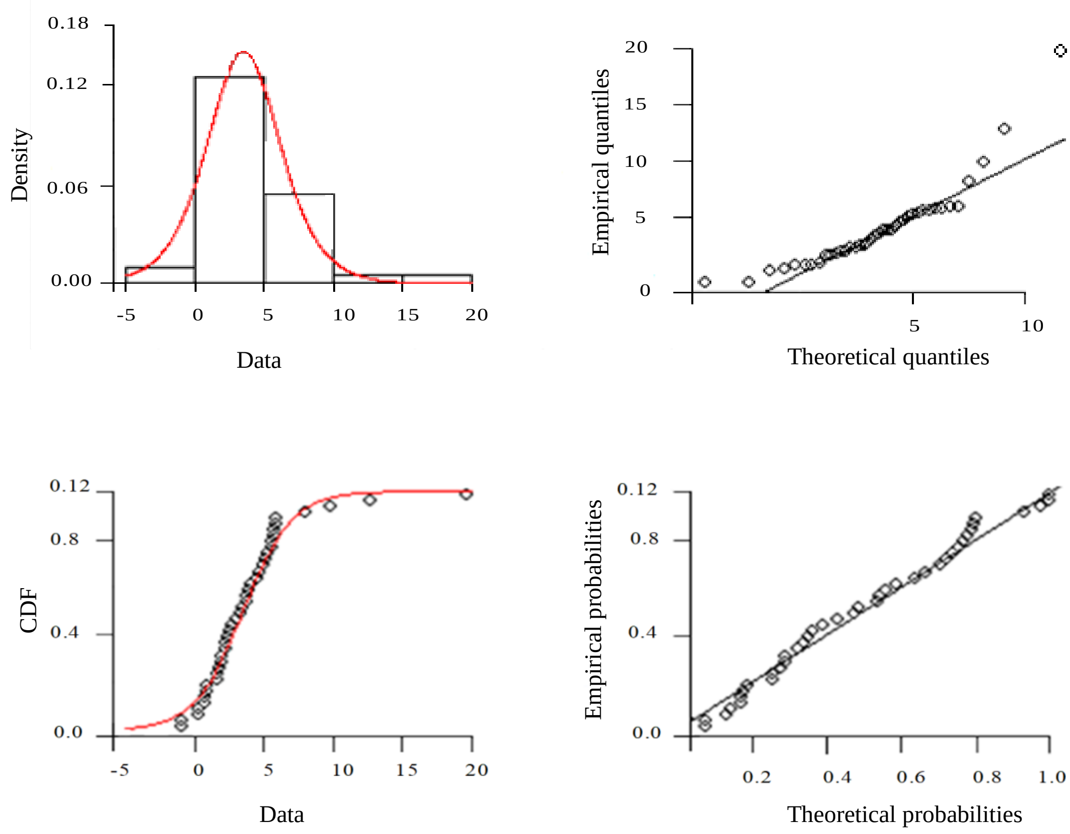

Figure 1 Distribution of the inflation rate data of the Thai economy.

Table 1 Estimated parameters and goodness-of-fit test of the Thai economy inflation rate data.

Fitting of the logistic distribution by maximum likelihood	
Location parameter	Scale parameter
$\mu = 3.472884$	$s = 1.770586$
Goodness of fit test for Logistic distribution of inflation rate: alternative hypothesis: two-sided	
One-sample Kolmogorov-Smirnov test	Data: inflationrate
D = 0.099769	p-value = 0.8208

From the results, the Thai economy inflation rate data follow a logistic distribution with location and scale parameters of 3.472884 and 1.770586, respectively; D statistic = 0.099769; and *p*-value = 0.8208 (> 0.05).

An example of evaluating the eigenvalue and eigenvector of the in-control state ARL is shown in matrix form as

$$A = \frac{1}{0.78}\begin{bmatrix} 443836.26 & 216.74 & -92535.95 & -6.70 & 7534.81 & -288.99 & 4.47 \\ 9.45*10\text{^}6 & 3994.21 & -1.90*10\text{^}6 & -114.38 & 147656.65 & -5325.61 & 76.25 \\ 9.45*10\text{^}6 & 3994.21 & -1.90*10\text{^}6 & -114.38 & 147656.65 & -5325.61 & 76.25 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1.06*10\text{^}11 & 3.47*10\text{^}7 & -1.99*10\text{^}10 & -865302.99 & 1.43*10\text{^}9 & -4.63*10\text{^}7 & 576868.66 \end{bmatrix}$$

The eigenvalues corresponding to the eigenvectors are reported in Table 2.

Table 2 Eigenvalue and eigenvector for in- control state ARL.

Eigenvalue	Eigenvector
$\kappa_1 = 41677.91$	7.29465*10^-6, 0.000118397, 0.000118397, 0.00220538, 0.00220538, 0.045451, 0.998962
$\kappa_2 =$ -40949.34	1.51507*10^-6, -0.000026518, -0.000026518, -0.0012991, -0.0012991, -0.0383207, -0.999264
$\kappa_3 =$ -2814.56	0.0000145917, 0.000240795, 0.000240795, 0.00393684, 0.00393684, 0.0635765, 0.997961
$\kappa_4 = 1646.34$	0.0000245903, 0.000240097, 0.000240097, 0.00106258, 0.00106258, -0.0268136, -0.999639
$\kappa_5 = 451.66$	-0.0000196512, -0.000312839, -0.000312839, -0.00479758, -0.00479758, -0.0706123, -0.997481
$\kappa_6 =$ -2.48733*10^-10	6.63733*10^-13, -0.727744, 1.08345*10^-11, -0.230373, 1.71486*10^-10, -0.545808, -0.34556
$\kappa_7 =$ 2.41868*10^-12	7.00363*10^-16, -0.00090481, 1.1231*10^-14, -0.5547, 1.74188*10^-13, -0.000678607, -0.83205

Furthermore, the eigenvalues and corresponding ARLs based on the parameters of inflation rate data of the Thai economy (Table 1) are summarized in Tables 3 and 4.

Table 3 Eigenvalue and its ARL for shift sizes 0.0, 0.1, and 0.3.

$\delta = 0.0$		$\delta = 0.1$		$\delta = 0.3$	
Eigenvalue	ARL0	Eigenvalue	ARL1	Eigenvalue	ARL1
41677.9	2867.85	41697.3	2847.6	41734.4	2807.19
-40949.3	3395.59	-40980.9	3365.79	-41040.9	3306.66
-2814.56	2.68033	-2810.82	2.61646	-2803.72	2.49236
1646.34	2.52203	1657.07	2.40732	1677.39	2.27021
451.66	2.19775	448.774	2.17853	443.416	2.14215
-2.48733*10^-10	1	1.52216*10^-10	1	-7.34585*10^-10	1
2.41868*10^-12	1	-2.73307*10^-12	1	-6.74905*10^-13	1

Table 4 Eigenvalue and its ARL for shift sizes 0.5, 0.7, and 0.9.

$\delta = 0.5$		$\delta = 0.7$		$\delta = 0.9$	
Eigenvalue	ARL1	Eigenvalue	ARL1	Eigenvalue	ARL1
41769.0	2766.93	41801.3	2726.82	41831.1	2686.88
-41096.9	3248.18	-41148.8	3190.33	-41196.7	3133.11
-2797.13	2.37288	-2791.05	2.25778	-2785.46	2.14683
1696.21	2.227641	1713.56	2.155668	1729.47	2.12792
438.567	2.1083	434.189	2.07675	430.249	2.04728
1.22529*10^-9	1	-5.32924*10^-10	1	-1.40895*10^-10	1
2.88138*10^-12	1	4.53515*10^-13	1	2.25506*10^-12	1

Tables 3 and 4 show the ARLs and eigenvalue of the inflation rate data of Thailand for $\mu = 3.472884,$ $s = 1.770586,$ $M = 3.3,$ $N = 10,$ and $\lambda = 0.11$. The results show that the ARL eigenvalues for the EWMA control chart of the inflation rate data of the Thai economy are all real and the maximum eigenvalue returns the maximum ARL0 and ARL1 values. Moreover, an eigenvalue close to zero returns ARL0 and ARL1 close to one and, when the shift size increases, the ARL decreases for all eigenvalues.

In this research, we offer an alternative for evaluating the ARL of a control chart. The eigenvalue is an important parameter for determining the direction of the solution of integral equations representing the ARL. The advantage of this method is that it can quickly detect an out-of-control signal because the ARL is decreased and tends to one when the eigenvalues approach zero. Moreover, for real applications, the probability density function of the observations is complex whereas this method can be carried out numerically based on the computation of the matrix. In future research, we will focus on the analytical ARL for the Thai economy inflation rate data or other economic applications such as exchange rate, income level, GDP, etc. Other control charts can also be considered and their performances with our method compared with the EWMA control chart. A limitation of this study is that the analytical solution for the integral equation cannot be carried out due to the complexity of the kernel on the integral equation.

4. Conclusion

Statistical process control, a very important concept for quality control of products and services, can also be applied to economic problems. The objectives of this research were to study integral equations and their eigenvalues to represent the ARL of an EWMA control chart when applied to inflation rate data of the Thai economy. Estimating the parameters and fitting the probability density function for the data were based on the maximum likelihood function and the Kolmogorov-Smirnov method, respectively. The results show that the appropriate probability density function of Thailand's inflation rate is for a logistic distribution with location and scale parameters $\mu = 3.472884$ and $s = 1.770586$, respectively. The method of ARL evaluation on an EWMA control chart was based on numerical integral equations via kernel approximation with a Taylor series and degenerating the kernel to a symmetric one in the integral equation. Moreover, the findings show that the maximum eigenvalue returns the maximum ARL0 and ARL1, while an eigenvalue close to zero returns ARL0 and ARL1 close to one. In addition, the approximate ARL converges and has an upper bound if $0 < \vartheta < 1$. Distinct eigenvalues of the ARL of an EWMA control chart of the Thai economy inflation rate data with a symmetric kernel are all real.

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