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More Fun with Pythagoras

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Abstract

Students may be introduced to a set of formulae that defines all primitive Pythagorean triples: For all positive integers *i* and *j* where *i* is an uneven number and *j* is an even number and *i* and *j* have no common factors, $h = i^2 + ij + j^2/2$, $e = ij + j^2/2$, $u = i^2 + ij$ where *h* is the hypotenuse, *e* is the even-numbered leg and *u* is the uneven-numbered leg of the Pythagorean triangle. A collection of well-defined subsets of the universal set of triangles described by *i* and *j* form sequences of triangles that approach in proportion a triangle which has one side defined by an irrational number while the other two are positive integers. This also creates sequences of rational numbers that approach an irrational limit. Finding and defining these series of triangles and numbers require the use of good algebra, giving relevance to the student's learning of factorising and manipulating algebraic expressions.

Introduction

As part of the Mathematics curriculum of most high schools across the world, pupils are introduced to the theorem of Pythagoras. It provides an interesting opportunity in algebra and geometry, where the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the legs of the triangle. As model triangle, the 3,4,5-triangle is used most often, and now and then the 5,12,13- and the 15,8,17-triangles also appear.

This paper provides an infinite matrix of Pythagorean triangles[1] for pupils, which may be generated by indices related to the hypotenuse (*h*), the even (*e*) and the uneven (*u*) legs with very simple algebra, well within reach of the high school pupil. From this matrix of triangles, infinite series of triangles may be selected that in their proportions approach a triangle that has one side irrational with respect to the other two sides.[2] From these series of triangles, series of rational numbers are generated that also have as their limit an irrational number. These are all concepts that are taught in high school mathematics, thus these series of triangles and rational numbers provide an opportunity for pupils to find application for some of these concepts.

These concepts also require rigorous algebra of high school standard because in converting from the formulae of the indices of these triangles to the formulae for respective sides of the triangles, algebra is required that includes factorisation, dealing with differences of squares, radicands and their manipulation, etc. The pupil is thus exposed to good algebra and geometry, and at the same time sees that all this mathematics has some kind of application.

Often the schooling systems across the world have a tendency to discourage original thought. This paper illustrates how children can be stimulated to pursue their own ideas and be creative, using the tools they acquire at school. Indeed, mathematics can be fun, even for children, and children that have been stimulated at school to have fun, turn out to be original thinkers, becoming engineers, architects, businessmen, politicians, and a whole lot of other vocations where creativity is required.

History

The author would like to share a little of the history of this Pythagorean mathematics, which in itself will model to the pupils how an idea and creative thinking came to fruition in providing the concepts of this paper.

As a primary school youngster at the age of around 10, I was playing with a Meccano set, making structures with right angles that had to be stabilised with diagonal girders. My grandfather (a carpenter by profession but very interested in mathematics) introduced me to Pythagoras' theorem to help me build structures with right angles. I was very fascinated by Pythagoras' theorem. My father (a primary school teacher who taught *ia* mathematics) showed me a mathematical concept in which the sequence of perfect squares was lined up. Their differences formed a new sequence that was all the uneven numbers. The differences of the differences was therefore 2.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361
dif	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	
dif2		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	

Bringing together what my grandfather had taught me, and my father had shown me, it dawned on me that if the differences of perfect squares was the sequence of all the odd numbers, then it quite clearly the perfect squares of all the odd numbers are a part of the sequence of differences. All these situations where the difference of two perfect squares was also a perfect square, each created a new Pythagorean triangle. For example, 9 is the difference between 5^2 and 4^2, and is in itself 3^2, therefore (3,4,5) is a Pythagorean triple. Similarly, 25 is the difference between 13^2 and 12^2, and is in itself 5^2, therefore (5,12,13) is a Pythagorean triple, etc. This provided an infinite series of Pythagorean triangles where the hypotenuse and the even leg differed by 1. Fibonacci[3,4] and Stifel[5,6] were aware of this infinite sequence of triangles. In a similar manner, I discovered the infinite series of triangles where the hypotenuse and the uneven leg differ by 2. This series has also already been described by Ozanam.[7]

I then went about seeking a series of triangles where the legs differed by only 1, and as a high school pupil, found it as well: {(3,4,5), (21,20,29), (119,120,169), (697,696,985), ……} An interesting aspect of this series of right-angled triangles is that the shape of the triangle more and more resembles the 45° right-angled triangle as the series progresses. Using this sequence of triangles, it is quite obvious that a series of rational numbers derived from dividing the hypotenuse with either of the legs may be produced, that have as their limit, an irrational number.

This is where my childhood exploration of right-angled triangles stopped, but since, I had wondered whether it would be possible to find other sequences of triangles that would approximate other non-rational right-angled triangles such as the 30/60° right-angled triangle. In the past five years I have had the opportunity to discover such series.

The indices

All primitive right-angled triangles are defined by a pair of indices (i,j) where i is an uneven positive integer, and j is an even positive integer, on condition the i and j do not have any common factors.[1] (If they do have a common factor, then the triangle will not be primitive.) The hypotenuse (h), the uneven leg (u) and the even leg (e) of the right-angled triangle are defined as follows:

$$u = i^2 + ij \qquad (1)$$

$$e = ij + j^2/2 \qquad (2)$$

$$h = i^2 + ij + j^2/2 \qquad (3)$$

This provides an infinite two-dimensional matrix of right-angled triangles which form the universal set from which the series of triangles are selected (Scheme 1).

The series of triangles were manually searched until a pattern was observed.[2] Once a pattern was identified for finding the next member of the series, the series was extended algebraically. To illustrate this concept, the series for the 45° triangle is used. The series is provided above and the formula for finding the next member of a series is:

For the series $(i, j)_n$, $(i, j)_{n+1} = (i_n + j_n, 2i_n + j_n)$ where $(i, j)_1 = (1, 2)$ (4)

Series in terms of the sides of triangles

Now converting this series of indices into the triangles they represent, requires combining what we have in terms of the indices, and converting that to h, u and e. This demands command of good algebra from the pupil, but also shows the pupil good application of many of the tools he has acquired in algebra. Continuing with the 45°-triangle the algebraic process is illustrated:

$$
\begin{aligned}
u_{n+1} &= i_{n+1}^2 + i_{n+1}j_{n+1} && \text{from (1)}\\
&= (i_n + j_n)^2 + (i_n + j_n)(2i_n + j_n) && \text{from (4)}\\
&= i_n^2 + 2i_nj_n + j_n^2 + 2i_n^2 + 3i_nj_n + j_n^2 && \\
&= 3i_n^2 + 5i_nj_n + 2j_n^2 && \text{(5)}\\
\text{However } h &= i^2 + e && \text{(3) and (2)}\\
\therefore i &= \sqrt{h-e} && \text{(6)}\\
\text{and } e &= ij + j^2/2 && \text{(2)}\\
\therefore 2e &= 2j\sqrt{h-e} + j^2 && \text{from (6)}\\
\therefore 0 &= j^2 + 2j\sqrt{h-e} - 2e && \\
\therefore j &= \frac{-2\sqrt{h-e}+\sqrt{4(h-e)+8e}}{2} && \text{roots of a quadratic equation}\\
&= \frac{-2\sqrt{h-e}+2\sqrt{h-e+2e}}{2} && \\
&= \sqrt{h+e} - \sqrt{h-e} && \text{(7)}\\
\therefore u_{n+1} &= 3\left(\sqrt{h_n-e_n}\right)^2 + 5\sqrt{h_n-e_n}\cdot\left(\sqrt{h_n+e_n}-\sqrt{h_n-e_n}\right) + 2\left(\sqrt{h_n+e_n}-\sqrt{h_n-e_n}\right)^2 && \text{(5), (6) and (7)}\\
&= 3h_n - 3e_n + 5\left(\sqrt{h_n^2-e_n^2} - h_n + e_n\right) + 2\left(h_n + e_n - 2\sqrt{h_n^2-e_n^2} + h_n - e_n\right) && \\
&= 3h_n - 3e_n + 5(u_n - h_n + e_n) + 2(2h_n - 2u_n) && \text{by Pythagoras}\\
&= 2h_n + 2e_n + u_n && \text{(8)}\\
e_{n+1} &= j_{n+1}^2/2 + i_{n+1}j_{n+1} && \text{from (2)}\\
\therefore 2e_{n+1} &= (2i_n + j_n)^2 + 2(i_n + j_n)(2i_n + j_n) && \text{from (4)}\\
&= 4i_n^2 + 4i_nj_n + j_n^2 + 4i_n^2 + 6i_nj_n + 2j_n^2 && \\
&= 8i_n^2 + 10i_nj_n + 3j_n^2 && \\
&= (4i_n + 3j_n)(2i_n + j_n) && \text{(9)}\\
&= \left[4\sqrt{h_n-e_n} + 3\left(\sqrt{h_n+e_n}-\sqrt{h_n-e_n}\right)\right]\left(2\sqrt{h_n-e_n} + \sqrt{h_n+e_n} - \sqrt{h_n-e_n}\right) && \text{from (6) and (7)}\\
&= \left(\sqrt{h_n-e_n} + 3\sqrt{h_n+e_n}\right)\left(\sqrt{h_n-e_n} + \sqrt{h_n+e_n}\right) && \\
&= h_n - e_n + 4u_n + 3h_n + 3e_n && \text{by Pythagoras and diff. of squares}\\
&= 4h_n + 2e_n + 4u_n && \\
\therefore e_{n+1} &= 2h_n + e_n + 2u_n && \text{(10)}\\
h_{n+1} &= i_{n+1}^2 + i_{n+1}j_{n+1} + j_{n+1}^2/2 && \text{from (3)}\\
&= e_{n+1} + i_{n+1}^2 && \text{from (2)}\\
2h_{n+1} &= 8i_n^2 + 10i_nj_n + 3j_n^2 + 2i_n^2 + 4i_nj_n + 2j_n^2 && \text{from (4) and (9)}\\
&= 10i_n^2 + 14i_nj_n + 5j_n^2 && \text{(11)}\\
&= 10\left(\sqrt{h_n-e_n}\right)^2 + 14\sqrt{h_n-e_n}\cdot\left(\sqrt{h_n+e_n}-\sqrt{h_n-e_n}\right) + 5\left(\sqrt{h_n+e_n}-\sqrt{h_n-e_n}\right)^2 && \text{from (6) and (7)}\\
&= 10h_n - 10e_n + 14\left(\sqrt{h_n^2-e_n^2} - h_n + e_n\right) + 5\left(h_n + e_n - 2\sqrt{h_n^2-e_n^2} + h_n - e_n\right) && \\
&= 10h_n - 10e_n + 14(u_n - h_n + e_n) + 5(2h_n - 2u_n) && \text{by Pythagoras}\\
&= 6h_n + 4e_n + 4u_n && \\
\therefore h_{n+1} &= 3h_n + 2e_n + 2u_n && \text{(12)}
\end{aligned}
$$

The algebra may seem quite formidable, but all the steps are within the reach of high school mathematics, maybe just a little long. After an exercise as long as this, it is useful to test the newly generated formulae by applying them to the already established series of triangles. For example, tested at the first level, we generate the second triangle from (3,4,5):

$$\begin{aligned} u_2 &= 2h_1 + 2e_1 + u_1 && \text{from (8)} \\ &= 10 + 8 + 3 \\ &= 21 \\ e_2 &= 2h_1 + e_1 + 2u_1 && \text{from (10)} \\ &= 10 + 4 + 6 \\ &= 20 \\ h_2 &= 3h_n + 2e_n + 2u_n && \text{from (12)} \\ &= 15 + 8 + 6 \\ &= 29 \end{aligned}$$

Indeed, we have reached the second member of the series of triangles, and therefore know that the algebra was developed without a hitch.

The algebraic formula to find the series of triangles that lead to the 30/60°-triangle follows:

For the series $(i, j)_n$, $(i, j)_{n+1} = (i_n + j_n, 2i_n + 3j_n)$ where $(i, j)_1 = (1, 2)$ (13)

The pupil may want to try his own hand at finding formulae to generate the next triangle in terms of the sides of the triangle, deriving it from the indices as was done above for the 45°-triangle.

Other triangles

Besides the 45°- and the 30/60°-triangles, there are many more. Table 1 illustrates a sequence of triangles that are the limits of series of primitive Pythagorean triangles. Note that the square roots of 9 (3) and of 25 (5) do not feature in these series because these numbers are not irrational, yet they are displayed in the table to provide continuity to the sequence of irrational triangles:

Table 1: A sequence of triangles with one side irrational.

hypotenuse	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
rational leg	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
irrational leg	√3	√5	√7	3	√11	√13	√15	√17	√19	√21	√23	5	√27	√29	√31	√33

The pupil may want to try a few more. Below are the formulae for finding the series of triangles in terms of the indices. Try converting the indices to *h*, *u* and *e*.

For the √5 triangles $(i, j)_n$, $(i, j)_{n+1} = (i_n + 2j_n, 2i_n + 3j_n)$ where $(i, j)_1 = (1, 2)$

For the √7 triangles $(i, j)_n$, $(i, j)_{n+1} = (5i_n + 9j_n, 6i_n + 11j_n)$ where $(i, j)_1 = (1, 2)$

For the √11 triangles $(i, j)_n$, $(i, j)_{n+1} = (7i_n + 15j_n, 6i_n + 13j_n)$ where $(i, j)_1 = (3, 2)$

For the √13 triangles $(i, j)_n$, $(i, j)_{n+1} = (13i_n + 30j_n, 10i_n + 23j_n)$ where $(i, j)_1 = (3, 2)$

For the √15 triangles $(i, j)_n$, $(i, j)_{n+1} = (3i_n + 7j_n, 2i_n + 5j_n)$ where $(i, j)_1 = (3, 2)$

For the √17 triangles $(i, j)_n$, $(i, j)_{n+1} = (3i_n + 8j_n, 2i_n + 5j_n)$ where $(i, j)_1 = (3, 2)$

For the √19 triangles $(i, j)_n$, $(i, j)_{n+1} = (131i_n + 351j_n, 78i_n + 209j_n)$ where $(i, j)_1 = (3, 2)$

For the √21 triangles $(i, j)_n$, $(i, j)_{n+1} = (43i_n + 120j_n, 24i_n + 67j_n)$ where $(i, j)_1 = (3, 2)$

For the √23 triangles $(i, j)_n$, $(i, j)_{n+1} = (19i_n + 55j_n, 10i_n + 29j_n)$ where $(i, j)_1 = (3, 2)$

Some of these series operate on big coefficients and will require extra vigilance to not make mistakes, but may all be verified by testing as illustrated above.

Series of rational numbers

Having established the series of triangles, choice of the right pair of sides will lead to a series of numbers that will converge to an irrational number as the limit. The 45º-triangle will again serve as an example. The 45º-triangle may be represented as having the two legs equal to 1, then the hypotenuse is. Taking the hypotenuse and the even leg of every triangle in this series as a rational number where the hypotenuse is the numerator and the even leg the denominator, we have a series of numbers that approach the √2 as their limit:

$$\left\{\frac{5}{4}, \frac{29}{20}, \frac{169}{120}, \frac{985}{696}, \dots \dots\right\}$$

Turning these rational numbers into decimal fractions, notice how they are approaching the approximated decimal value for the √2 (1.4142135):

{1.20, 1.45, 1.408333, 1.4152298, ……}

The pupil may apply this same procedure for √3, √5, etc. Just bear in mind that for √5 and upward, to get the approximation of the root, the even side has to be multiplied by a factor in the numerator. This is because none of the sides in the triangle for √5 are one but a bigger integer. For √5 the formula is 2e/u, derived from the 2,√5,3-triangle (see Table 1). Likewise, for √7 the formula is 3e/u, derived from the 3,√7,4-triangle, etc.

Conclusion

New developments in the classical field of mathematics with respect to Pythagorean triangles have opened new avenues in the training of mathematics at the upper high school level. Pythagorean triangles form an infinite two dimensional matrix defined by indices (i, j) where i is an uneven positive integer and j is an even positive integer. Pupils can use these indices to find new triangles by substituting numbers for the indices and determining the lengths of the respective sides of the triangles.

Series of triangles may be found in the infinite matrix that have as their limiting triangles, a triangle with one side irrational. For example, for the 45º-triangle:

the series $(i, j)_n$; $(i, j)_{n+1} = (i_n + j_n, 2i_n + j_n)$ where $(i, j)_1 = (1, 2)$

and for the 30º/60º-triangle:

the series $(i, j)_n$, $(i, j)_{n+1} = (i_n + j_n, 2i_n + 3j_n)$ where $(i, j)_1 = (1, 2)$

Algebraic formulae define the $(n+1)^{th}$ triangle in the series where the n^{th} one is known (the 1^{st} triangle is defined) in terms of the indices. Formulae may then be algebraically developed to redefine the $(n+1)^{th}$ triangle in the series in terms of the sides of the n^{th} triangle. For example, for the 45º-triangle [the first triangle is (3,4,5)]:

$$u_{n+1} = 2h_1 + 2e_1 + u_1$$
$$e_{n+1} = 2h_n + e_n + 2u_n$$
$$h_{n+1} = 3h_n + 2e_n + 2u_n$$

This requires rigorous algebra, but well within the scope of high school algebra. The series of triangles in terms of the sides of triangles, provide an easy way to generate series of rational numbers that have as their limit an irrational number. For example, in the 45º-triangle, the square root of 2 may be found by using *h/u* or *h/e* of the series of triangles to define the respective series that will have as their limit the square root of 2.

This mathematics should boost the quality of the high school mathematical curriculum in that it provides meaningful application to many of the aspects of algebra. Pupils get an opportunity to see how mathematics may be applied in solving real problems. This will also provide the more gifted students with something that may grip their attention, generating interest in mathematics, and possibly opening doors to careers that are more mathematically inclined such as engineering, statistics, economics and other related occupations.

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